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Upper Set Rules with Binary Ranges*

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Abstract

We investigate the social choice problem in which the range of a rule consists of only two alternatives. We introduce the "positional relationship over the set of preferences", which partially describes the positions of two preferences. We show that a rule is "strategy-proof"iff it is "monotonic with respect to the positional relationship". In addition, we show that any *strategy-proof* rule is defined by an "upper set with respect to the positional relationship".

JEL Classification: D71

Key words: Upper set rules; Binary range; Strategy-proofness; Positional relationship; (x, y)-monotonicity

1 Introduction

We study the social choice problem in which the range of a rule consists of only two alternatives. The range of a rule can be binary in some environments (Example 1): problems involving choosing one alternative on the set of all single-peaked preferences in which the range of a rule has some disconnected jumps (Barberà and Jackson [5],

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Hagiwara et al. [8]) and problems of locating a public facility on the set of all single-dipped preferences (Manjunath [13]).[1] In the former problem, if the "minmax rule" selects an alternative in a disconnected jump, then an alternative is chosen by a "strategy-proof" rule whose range consists of the two extreme points in the jump.[2] In the latter problem, if a rule is *strategy-proof*, it can choose only the two extreme points in the range of the rule.[3]

In the social choice problem in which the range of a rule is $\{x, y\}$, Barberà et al. [3] capture the feature of *strategy-proof* rules by focusing on the "monotonicity condition with respect to the set of agents who prefer x to y". A rule is *essentially xy-monotonic* whenever, if x is chosen by the rule at a preference profile and some agents change their preferences so that the set of agents who prefer x to y is increasing, x is chosen by the rule at the new preference profile. Their characterization of *strategy-proof* rules with binary ranges is based on this monotonicity condition.[4] However, this characterization is "open", because rules do not describe how to select alternatives explicitly.[5]

We newly introduce an incomplete binary relationship over the set of preferences which

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[1] Our results also apply to the pure exchange economy (Barberà and Jackson [6]) in which the range of a rule has some disconnected jumps, the social choice problem in which there are only two alternatives (Harless [9], Lahiri and Pramanik [10], Larsson and Svensson [11], Marchant and Mishra [14], Manjunath [12]), and the problem of locating a public facility on the set of preference profiles such that each agent in a subset M of the set N of agents has single-peaked preferences and each agent in $N \backslash M$ has single-dipped preferences (Alcalde-Unzu and Vorsatz [1]).

[2] For the definition of the minmax rule, see, for example, Massò and Moreno de Barreda [15].

[3] For more detailed discussions concerning the range of a rule on the set of all single-dipped preferences, see Barberà et al. [4].

[4] In the problem of choosing a subset of a finite set of indivisible objects with strict preferences, Barberà et al. [7] define "voting by committees" based on a similar monotonicity condition to *essential xy-monotonicity*. In the social choice problem in which there are only two alternatives when agents may be indifferent between them, a similar monotonicity condition is also considered in "voting by extended committees" by Larsson and Svensson [11] and Manjunath [12].

[5] For the discussion concerning open and "closed" characterizations, see Massò and Moreno de Barreda [15]. They provide a closed characterization of *strategy-proof* rules in which they describe how to select alternatives explicitly in the problem involving choosing one alternative on the set of "symmetric" single-peaked preferences in which the range of a rule has some disconnected jumps. Applying our results, Hagiwara et al. [8] proposes a closed characterization of *strategy-proof* rules in this problem not only on the set of symmetric single-peaked preferences, but also "asymmetric" single-peaked preferences. In the same problem as ours, Barberà et al. [3] provide a closed characterization of "strongly" *strategy-proof* rules with binary ranges. For the definition of *strong strategy-proofness*, see Barberà et al. [3].

partially describes the positions of two preferences. We refer to this as the "positional relationship over the set of preferences". We consider the "monotonicity condition with respect to this positional relationship", instead of the monotonicity condition with respect to the set of agents who prefer x to y. A rule is (x, y)-*monotonic* whenever, if x is chosen by the rule at a preference profile and some agents change their preferences so that the ranking of x in $\{x, y\}$ is increasing, x is chosen by the rule at the new preference profile.

We show that a rule whose range is $\{x, y\}$ is *strategy-proof* iff it is (x, y)-*monotonic* (Theorem 1). This is an open characterization.[6] We propose a "closed"characterization in which rules describe how to select alternatives explicitly. We show that any *strategy-proof* rule whose range consists of only two alternatives is defined by an "upper set with respect to the positional relationship"(Theorem 2).[7]

This paper is organized as follows. Section 2 introduces the model and axioms of a rule. Section 3 provides definitions for the positional relationship over the set of preferences and *monotonicity with respect to the positional relationship*, and proposes a characterization of *strategy-proof* rules with binary ranges by *monotonicity.* Section 4 presents definitions for an upper set with respect to the positional relationship and an upper set rule as well as our main result.

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[6]In the same model as ours, Vannucci [17] provides an open characterization of *strategy-proof* rules with binary ranges. In the problem involving choosing one alternative on the set of single-peaked preferences in which the range of a rule has some disconnected jumps, Barberà and Jackson [5] also proposes an open characterization of *strategy-proof* rules with binary ranges.

[7]In a social choice problem with a binary range of a rule and strict preferences, *strategy-proof* rules are characterized (Rao et al. [16]). There are two differences between Rao et al. [16] and this paper. First, in their model, indifferences are not allowed, but we allow them. Second, in their characterization, a nonempty collection of agent coalitions is considered. In our characterization, however, we consider an upper set of a set of preferences with respect to the positional relationship.

2 The model and axioms

Let $N = \{1, ..., n\}$ be the **set of agents** and A be the **set of alternatives** such that $|A| \geq 2$. Let $\mathcal{R}$ be the **set of admissible preferences**. An element of $\mathcal{R}$ is denoted by R, whose asymmetric and symmetric components are denoted by P and I, respectively. If we consider a specific agent, $i \in N$, then we denote $R_i \in \mathcal{R}$, P_i, and I_i. Let $R^n = (R_1, ..., R_n)$ be a **preference profile** and $\mathcal{R}^n = \times_{i \in N} \mathcal{R}$. For each $i \in N$, let $R^n_{-i} = (R_1, ..., R_{i-1}, R_{i+1}, ..., R_n) \in \times_{j \neq i} \mathcal{R}$. For each $S \subset N$, let $R_S \in \times_{i \in S} \mathcal{R}$ and $R_{N \setminus S} \in \times_{i \in N \setminus S} \mathcal{R}$.

A **rule** is a single-valued mapping $f : \mathcal{R}^n \to A$ which, for each $R^n \in \mathcal{R}^n$, specifies $f(R^n) \in A$. We assume that the range of a rule consists of only two alternatives. Formally, $|\{f(R^n) \in A : R^n \in \mathcal{R}^n\}| = 2$. Hereafter, let $\{f(R^n) \in A : R^n \in \mathcal{R}^n\} = \{x, y\}$. We define the **positional relationship** $\leq_{(x,y)}$ **over** $\{x, y\}$ as follows:

$$a \leq_{(x,y)} b \Longleftrightarrow \begin{cases} a = x, b = y \\ a = x, b = x \\ a = y, b = y \end{cases}$$

Then, $\leq_{(x,y)}$ determines the position between x and y. Without a loss of generality, we consider that "x is located on the left side of y".

There are some economic environments in which the range of a rule consists of only two alternatives with at least two alternatives.

Example 1. We consider two economic environments: (1) problems involving choosing one alternative on the set of all single-peaked preferences when the range of a rule may have disconnected jumps, and (2) problems of locating a public facility on the set of all

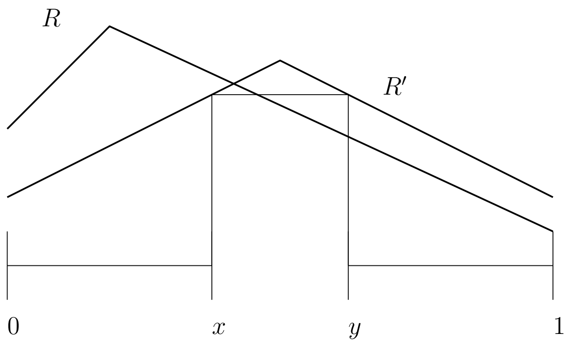

Figure 1. Single-peaked preferences when the range of a rule has disconnected jumps.

single-dipped preferences. Let $A = [0,1]$.

(1) Single-peaked preferences when the range of a rule has disconnected jumps:

Barberà and Jackson [5] and Hagiwara et al. [8] study *strategy-proof* rules on the set of all "single-peaked" preferences when the range of a rule may have disconnected jumps. A preference R is **single-peaked** on $[0,1]$ if there is the peak $p(R) \in [0,1]$ such that for each pair $a, b \in [0,1]$, if $b < a \leq p(R)$ or $p(R) \leq a < b$, then $a \; P \; b$.

If the "minmax rule" selects an alternative in a disconnected jump, then an alternative is chosen by a *strategy-proof* rule whose range consists of the two extremes of the jump (Figure 1).

(2) Single-dipped preferences:

Manjunath [13] studies *strategy-proof* rules when agents have "single-dipped" preferences. A preference R is **single-dipped** on $[0,1]$ if there is the dip $d(R) \in [0,1]$ such that for each pair $a, b \in [0,1]$, if $a < b \leq d(R)$ or $d(R) \leq b < a$, then $a \; P \; b$.

If a rule is *strategy-proof*, then it can choose only the two extreme points in the range of the rule (Figure 2).$\Diamond$

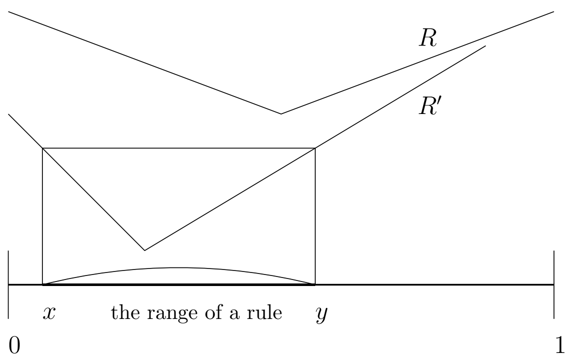

Figure 2. Single-dipped preferences.

The following are two strategic axioms of rule f. *Strategy-proofness* states that in the direct revelation game, for each agent, truth-telling is a dominant strategy.

Strategy-proofness: For each $R^n \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$,

$$f(\overset{\text{truth}}{R_i}, R^n_{-i}) \overset{\text{truth}}{R_i} f(\overset{\text{lie}}{R'_i}, R^n_{-i}).$$

"Group strategy-proofness" requires that no group of agents should ever be able to make its members better off by jointly misrepresenting their preferences.

Group strategy-proofness: For each $R^n \in \mathcal{R}^n$, each $S \subseteq N$, and each $R'_S \in \times_{i \in S}\mathcal{R}$, there is $i \in S$ such that

$$f(\overset{\text{truth}}{R_S}, R_{N\setminus S}) \overset{\text{truth}}{R_i} f(\overset{\text{lie}}{R'_S}, R_{N\setminus S}).$$

The following is applied in our result.

Proposition 1. (Barbera et al. [2]) *Let f be a rule whose range consists of only two alternatives. Then, f is* strategy-proof *iff it is* group strategy-proof.

3 Monotonicity and strategy-proofness

In order to capture the feature of *strategy-proof* rules whose ranges consist of only two alternatives, imposing the following positional structure over $\mathcal{R}$ is useful.

Let $\mathcal{R}_x = \{R \in \mathcal{R} : x \ P \ y\}$, $\mathcal{R}_y = \{R \in \mathcal{R} : y \ P \ x\}$, and $\mathcal{R}_{xy} = \{R \in \mathcal{R} : x \ I \ y\}$. That is, $\mathcal{R}_x$ is the **set of preferences in which x is preferred to y**, $\mathcal{R}_y$ is the **set of preferences in which y is preferred to x**, and $\mathcal{R}_{xy}$ is the **set of preferences in which x and y are indifferent**.

We define the positional relationship $\precsim_{(x,y)}$ over $\mathcal{R}$, whose symmetric and asymmetric parts are denoted by $\prec_{(x,y)}$ and $\sim_{(x,y)}$, respectively, as follows.

Positional relationship $\precsim_{(x,y)}$ over $\mathcal{R}$: For each pair $R, R' \in \mathcal{R}$,

$$R \prec_{(x,y)} R' \iff \begin{cases} R \in \mathcal{R}_x \text{ and } R' \in \mathcal{R}_{xy} \\ R \in \mathcal{R}_x \text{ and } R' \in \mathcal{R}_y \\ R \in \mathcal{R}_{xy} \text{ and } R' \in \mathcal{R}_y \end{cases}$$

$$R \sim_{(x,y)} R' \iff \begin{cases} R, R' \in \mathcal{R}_x \\ R, R' \in \mathcal{R}_y \\ R, R' \in \mathcal{R}_{xy} \text{ and } R = R'. \end{cases}$$

Note that $\precsim_{(x,y)}$ satisfies *reflexivity* and *transitivity*, but not *completeness*. For *incompleteness*, we can easily check that for each pair $R, R' \in \mathcal{R}_{xy}$ such that $R \neq R'$, $\precsim_{(x,y)}$ imposes no relationship between R and R' (Figure 3).

We introduce an important property of rule f, (x, y)-*monotonicity*, to characterize the class of *strategy-proof* rules with binary ranges.

(x, y)-**monotonicity**: For each pair $R^n, R'^n \in \mathcal{R}^n$, if for each $i \in N$, $R_i \precsim_{(x,y)} R'_i$, then $f(R^n) \leq_{(x,y)} f(R'^n)$.

We interpret (x, y)-*monotonicity* as follows: if an agent, $i \in N$, only changes his

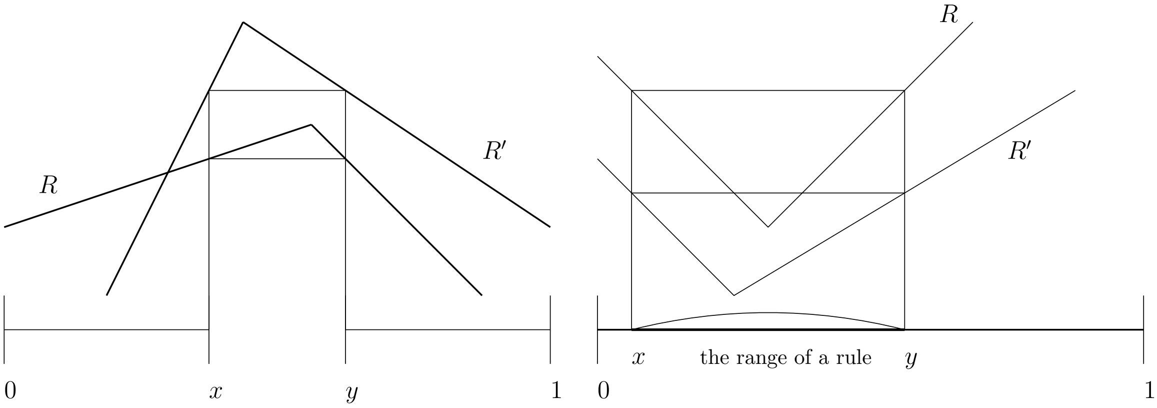

Figure 3. *Incompleteness.*

preference R_i to R'_i such that $R_i \precsim_{(x,y)} R'_i$ and the outcome of a rule at (R_i, R^n_{-i}) is not same as the outcome of the rule at (R'_i, R^n_{-i}), the outcome can move only in the direction from x to y. Formally, if $R_i \precsim_{(x,y)} R'_i$ and $f(R_i, R^n_{-i}) \neq f(R'_i, R^n_{-i})$, then $f(R_i, R^n_{-i}) = x$ and $f(R'_i, R^n_{-i}) = y$.

The following is an open characterization of *strategy-proof* rules with binary ranges.

Theorem 1. *Let f be a rule whose range is $\{x, y\}$. Then, f is* strategy-proof *iff it is* (x, y)-monotonic.

Proof.

Necessity: Let f be a *strategy-proof* rule. We prove it by contradiction. Suppose that f is not (x, y)-*monotonic*. Then, there are $R^n \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $R_i \precsim_{(x,y)} R'_i$, $f(R_i, R^n_{-i}) = y$, and $f(R'_i, R^n_{-i}) = x$. There are six cases concerning $R_i \precsim_{(x,y)} R'_i$.

Case 1. $R_i \in \mathcal{R}_x$ and $R'_i \in \mathcal{R}_{xy}$.

If the true preference profile is R^n, then $f(R'_i, R^n_{-i}) = x \ P_i \ y = f(R_i, R^n_{-i})$, so that f violates *strategy-proofness*, which is a contradiction.

Case 2. $R_i \in \mathcal{R}_x$ and $R'_i \in \mathcal{R}_y$.

If the true preference profile is R^n, then $f(R'_i, R^n_{-i}) = x \; P_i \; y = f(R_i, R^n_{-i})$, so that f violates *strategy-proofness*, which is a contradiction.

Case 3. $R_i \in \mathcal{R}_{xy}$ and $R'_i \in \mathcal{R}_y$.

If the true preference profile is (R'_i, R^n_{-i}), then $f(R_i, R^n_{-i}) = y \; P'_i \; x = f(R'_i, R^n_{-i})$, so that f violates *strategy-proofness*, which is a contradiction.

Case 4. $R_i, R'_i \in \mathcal{R}_x$.

If the true preference profile is R^n, then $f(R'_i, R^n_{-i}) = x \; P_i \; y = f(R_i, R^n_{-i})$, so that f violates *strategy-proofness*, which is a contradiction.

Case 5. $R_i, R'_i \in \mathcal{R}_y$.

If the true preference profile is (R'_i, R^n_{-i}), then $f(R_i, R^n_{-i}) = y \; P'_i \; x = f(R'_i, R^n_{-i})$, so that f violates *strategy-proofness*, which is a contradiction.

Case 6. $R_i, R'_i \in \mathcal{R}_{xy}$.

In this case, $R_i = R'_i$. Hence, there is no pair $R_i, R'_i \in \mathcal{R}_{xy}$ such that $R_i \precsim_{(x,y)} R'_i$ and $f(R_i, R^n_{-i}) \neq f(R'_i, R^n_{-i})$.

Sufficiency: Let f be a (x, y)-*monotonic* rule. We prove it by contradiction. Suppose that f is not *strategy-proof*. By Proposition 1, f is not *group strategy-proof*. Then, there are $R^n \in \mathcal{R}^n$, $S \subseteq N$, and $R'_S \in \times_{i \in S} \mathcal{R}$ such that for each $i \in S$, $f(R'_S, R_{N \setminus S}) \; P_i \; f(R_S, R_{N \setminus S})$. There are two cases concerning $f(R_S, R_{N \setminus S})$ and $f(R'_S, R_{N \setminus S})$.

Case 1. $f(R_S, R_{N \setminus S}) = x$ and $f(R'_S, R_{N \setminus S}) = y$.

Since for each $i \in S$, $y \; P_i \; x$, we have $S \subseteq \{i \in N : R_i \in \mathcal{R}_y\}$. Then, by the definition of $\precsim_{(x,y)}$, for each $i \in S$ and each $R'_i \in \mathcal{R}$, we have $R'_i \precsim_{(x,y)} R_i$ regardless of R'_i. Since $f(R'_S, R_{N \setminus S}) = y$, (x, y)-*monotonicity* implies $f(R_S, R_{N \setminus S}) = y$, which is a contradiction.

Case 2. $f(R_S, R_{N\setminus S}) = y$ and $f(R'_S, R_{N\setminus S}) = x$.

Since for each $i \in S$, $x \; P_i \; y$, we have $S \subseteq \{i \in N : R_i \in \mathcal{R}_x\}$. Then, by the definition of $\precsim_{(x,y)}$, for each $i \in S$ and each $R'_i \in \mathcal{R}$, we have $R_i \precsim_{(x,y)} R'_i$ regardless of R'_i. Since $f(R_S, R_{N\setminus S}) = y$, (x, y)-*monotonicity* implies $f(R'_S, R_{N\setminus S}) = y$, which is a contradiction.

■

4 Upper set rules

We provide a "closed" characterization of *strategy-proof* rules with binary ranges in which they describe how to select alternatives explicitly. For each $X \subseteq \mathcal{R}^n$, let

$$U(X) \equiv \{R^n \in \mathcal{R}^n : \text{ there is } R'^n \in X \text{ such that for each } i \in N, \; R'_i \precsim_{(x,y)} R_i\}.$$

A subset $\mathcal{D}$ of $\mathcal{R}^n$ is an **upper set with respect to** $\precsim_{(x,y)}$ if $\mathcal{D} = U(\mathcal{D})$. Let $\tilde{\mathcal{D}}$ be the **family of upper sets**.

We define an upper set rule as follows:

Upper set rule: There is $\mathcal{D} \in \tilde{\mathcal{D}}$ such that for each $R^n \in \mathcal{R}^n$,

$$g^{\mathcal{D}}(R^n) = \begin{cases} x & \text{if } R^n \notin \mathcal{D} \\ y & \text{if } R^n \in \mathcal{D}. \end{cases}$$

Similar to Massò and Moreno de Barreda [15] and Rao et al. [16], there are several such mappings because there are several upper sets. The following example illustrates the point.

Example 2. There are two agents, $N = \{1, 2\}$. For each $i \in N$, let $R_i \in \mathcal{R}_{xy}$ and

$R'_i \in \mathcal{R}_y$. Let $\mathcal{R}^n = \{(R_1, R_2), (R_1, R'_2), (R'_1, R_2)\}$.

In this setting, if $\mathcal{D} = \{(R_1, R_2)\}$, then $U(\mathcal{D}) = \{(R_1, R_2), (R_1, R'_2), (R'_1, R_2)\}$. Hence, $\mathcal{D} \neq U(\mathcal{D})$. If $\mathcal{D} = \{(R_1, R'_2)\}$, then $U(\mathcal{D}) = \{(R_1, R'_2)\}$. Hence, $\mathcal{D} = U(\mathcal{D})$. If $\mathcal{D} = \{(R'_1, R_2)\}$, then $U(\mathcal{D}) = \{(R'_1, R_2)\}$. Hence, $\mathcal{D} = U(\mathcal{D})$.

If $\mathcal{D} = \{(R_1, R_2), (R_1, R'_2)\}$, then $U(\mathcal{D}) = \{(R_1, R_2), (R_1, R'_2), (R'_1, R_2)\}$. Hence, $\mathcal{D} \neq U(\mathcal{D})$. If $\mathcal{D} = \{(R_1, R_2), (R'_1, R_2)\}$, then $U(\mathcal{D}) = \{(R_1, R_2), (R_1, R'_2), (R'_1, R_2)\}$. Hence, $\mathcal{D} \neq U(\mathcal{D})$. If $\mathcal{D} = \{(R_1, R'_2), (R'_1, R_2)\}$, then $U(\mathcal{D}) = \{(R_1, R'_2), (R'_1, R_2)\}$. Hence, $\mathcal{D} = U(\mathcal{D})$.

If $\mathcal{D} = \{(R_1, R_2), (R_1, R'_2), (R'_1, R_2)\}$, then $U(\mathcal{D}) = \{(R_1, R_2), (R_1, R'_2), (R'_1, R_2)\}$. Hence, $\mathcal{D} = U(\mathcal{D})$.

Therefore, $\tilde{\mathcal{D}} = \{\{(R_1, R'_2)\}, \{(R'_1, R_2)\}, \{(R_1, R'_2), (R'_1, R_2)\}, \{(R_1, R_2), (R_1, R'_2), (R'_1, R_2)\}\}$.$\Diamond$

The following is our main result.

Theorem 2. *Let f be a rule whose range is $\{x, y\}$. Then, f is* strategy-proof *iff it is an upper set rule.*

Proof.

Necessity: Let f be a *strategy-proof* rule. By Theorem 1, f is (x, y)-*monotonic*. Let $\mathcal{D}_f = \{R^n \in \mathcal{R}^n : f(R^n) = y\}$.

We prove $\mathcal{D}_f \in \tilde{\mathcal{D}}$ by contradiction. Suppose that $\mathcal{D}_f \notin \tilde{\mathcal{D}}$. Since $U(\mathcal{D}_f) \supseteq \mathcal{D}_f$ by the definition, $U(\mathcal{D}_f) \supsetneq \mathcal{D}_f$. Therefore, there is $R^n \in U(\mathcal{D}_f) \backslash \mathcal{D}_f$. Since $R^n \notin \mathcal{D}_f$, $f(R^n) = x$. Furthermore, since $R^n \in U(\mathcal{D}_f)$, there is $R'^n \in \mathcal{D}_f$ such that for each $i \in N$, $R'_i \precsim_{(x,y)} R_i$. By (x, y)-*monotonicity*, since $f(R^n) = x$, we have $f(R'^n) = x$. Then, $R'^n \notin \mathcal{D}_f$, which is a contradiction.

If $f(R^n) = x$, then $R^n \notin \mathcal{D}_f$ so that $g^{\mathcal{D}_f}(R^n) = x$. If $f(R^n) = y$, then $R^n \in \mathcal{D}_f$ so that $g^{\mathcal{D}_f}(R^n) = y$. That is, $f(R^n) = g^{\mathcal{D}_f}(R^n)$.

Sufficiency: Let $\mathcal{D} \in \tilde{\mathcal{D}}$ be such that for each $R^n \in \mathcal{R}^n$, $f(R^n) = g^{\mathcal{D}}(R^n)$. We prove it by contradiction. Suppose that f is not *strategy-proof*. Then, there are $R^n \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $f(R'_i, R^n_{-i})\ P_i\ f(R_i, R^n_{-i})$. By the definition of $g^{\mathcal{D}}$, $f(R_i, R^n_{-i})$ is equal to either x or y.

Case 1: $f(R_i, R^n_{-i}) = x$.

Since $f(R_i, R^n_{-i}) = x$ and $f(R'_i, R^n_{-i})\ P_i\ f(R_i, R^n_{-i})$, we have $R^n \notin \mathcal{D}$, $f(R'_i, R^n_{-i}) = y$, and $(R'_i, R^n_{-i}) \in \mathcal{D}$. On the other hand, since $y\ P_i\ x$, we have $R'_i \precsim_{(x,y)} R_i$ so that $R^n \in \mathcal{D}$, which is a contradiction.

Case 2: $f(R_i, R^n_{-i}) = y$.

Since $f(R_i, R^n_{-i}) = y$ and $f(R'_i, R^n_{-i})\ P_i\ f(R_i, R^n_{-i})$, we have $R^n \in \mathcal{D}$, $f(R'_i, R^n_{-i}) = x$, and $(R'_i, R^n_{-i}) \notin \mathcal{D}$. On the other hand, since $x\ P_i\ y$, we have $R_i \precsim_{(x,y)} R'_i$ so that $R^n \notin \mathcal{D}$, which is a contradiction.■

5 Conclusion

We investigated the social choice problem in which the range of a rule is $\{x, y\}$. We first specified the feature of *strategy-proof* rules: a rule is *strategy-proof* iff it is (x, y)-*monotonic* (Theorem 1). In this result, rules do not describe how to select alternatives explicitly. We then identified the class of *strategy-proof* rules explicitly: any *strategy-proof* rule is defined by an upper set with respect to the positional relationship (Theorem 2).

Our results can be applied to some environments as *strategy-proof* "tie-breaking rules". For example, in the problem involving choosing one alternative on the set of all single-peaked preferences in which the range of a rule has some disconnected jumps, Hagiwara et al. [8] provide a full characterization of the class of *strategy-proof* rules without any other axioms by applying our result.

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