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A CVaR model with an optimal hedging strategy for international portfolio selection

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Abstract For international portfolio selection, currency risks affect the performance of a portfolio. We can reduce currency risks by introducing forward contracts. In this paper, we propose an optimization model for international portfolio selection with hedging strategies using forward contracts. As a risk measure, we use Conditional Value at Risk (CVaR) of the portfolio's return. The model decides both a portfolio and hedge ratios simultaneously such that CVaR is minimized under the condition that the expected return of the portfolio is not less than a predetermined value. At first, the model is formulated as a nonlinear programming problem. Then we will show that the model is equivalently converted to a linear programming problem. We conduct numerical experiments to evaluate the proposed model. As a result, we observe that the proposed model automatically adjust the portfolio and the hedge ratios according to market environments and this adjustment leads to a stable performance of the portfolio.

Keywords: International portfolio selection, Currency risk, Currency hedging, CVaR

1. Introduction

Modern portfolio theories teach us the importance of the diversification of a portfolio. In that sense, it is natural to consider international diversified investment. However, in the case of international investment, we need to be careful about currency risks, since they greatly affect the performance of a portfolio. Effects of currency risks in international diversified investment are discussed in Eun and Resnick [5]. They propose to introduce forward contracts using short selling and conclude that portfolios with the hedging strategy almost always outperform unhedged portfolios. Glen and Jorion [4] confirm benefits of forward contracts in international diversified investment through statistical analysis using historical data. Topaloglou et al. [2] propose an optimization model for multi currency asset allocation with hedge strategies. In their model, each foreign currency is either fully hedged or not hedged at all.

In this paper, we propose an optimization model for international portfolio selection with hedging strategies using forward contracts. As a risk measure, we use Conditional Value at Risk (CVaR) of the portfolio's return. The model decides both a portfolio and hedge ratios simultaneously such that CVaR is minimized under the condition that the expected return of the portfolio is not less than a predetermined value. At first, the model is formulated as a nonlinear programming problem. Then we will show that the model is equivalently converted to a linear programming problem, which can be solved easily.

To evaluate the proposed model, we conduct numerical experiments based on actual market data. We compare the proposed model with models of no-hedging and full hedging strategies. As a result, we observe that the proposed model automatically adjusts the portfolio and the hedge ratios according to market environments and this adjustment leads to a stable performance of the portfolio.

This paper is organized as follows. In Section 2, we propose the CVaR model with an optimal hedging strategy. In Section 3, we show results of numerical experiments. Finally we conclude our paper in Section 4.

2. An optimization model for international portfolio selection

In this section, we construct an optimization model for finding a portfolio and hedge ratios simultaneously for international portfolio selection such that CVaR of a return is minimized under the condition that the expected return is not less than a predetermined value.

Suppose that we invest capital V (Japanese yen: JPY) in n assets (or stock indices) for a single period, where the asset $j \in \{1, 2, \ldots, n\}$ is dealt in a country j. For simplicity of discussion, we assume that all the countries are different and the country 1 is Japan. We call the currency in country j simply the currency j. For each $j \in \{1, \ldots, n\}$, let V_j (JPY) be the portion of the capital invested in the asset j. Let $x_j = V_j/V$ be the ratio of the capital invested in the asset j. Then we have $\sum_{j=1}^{n} x_j = 1$. In this paper, we do not consider short selling, thus $x_j \geq 0$ for each $j \in \{1, \ldots, n\}$. The vector $x = (x_1, \ldots, x_n)$ is called a portfolio.

In international portfolio selection, the return of the portfolio is greatly affected by foreign exchange rates. We use forward contracts to reduce the effect of the foreign exchange rates. By using a forward contract, we can fix the exchange rate at the end of the investment period. For each $j \in \{1, \ldots, n\}$, let s_j be the forward rate in the country j. This means that we can exchange one unit of the currency j by a fixed rate s_j at the end of the period. Let i_j and e_j for each $j \in \{1, \ldots, n\}$ be the interest rate for the investment period and the exchange rate at the beginning of the period in the country j, respectively. Then we use the following forward rate

$$s_j = \frac{1 + i_1}{1 + i_j} e_j.$$

Next we obtain a concrete formula for the return of the portfolio when we adopt the forward contracts. For each $j \in \{1, \ldots, n\}$, let P_j be the price of the asset j at the beginning of the period, where P_j is represented by the currency j. Since we invest V_j (JPY) in the asset j, we exchange it to V_j/e_j units of the currency j. Then we purchases $V_j/(e_j P_j)$ units of the asset j. Let $h_j \in [0, 1]$ be the hedge ratios for the currency j, that is, we make a forward contract of $h_j V_j/e_j$ units for the currency j at the forward rate s_j. Let P'_j and e'_j be the unknown price of the asset j and the unknown exchange rate of the currency j at the end of the investment period. By the investment of the asset j, we obtain $(P'_j V_j)/(e_j P_j)$ units of the currency j. For this amount, we exchange $h_j V_j/e_j$ units at the forward rate s_j and the rest at the rate e'_j. To summarize the discussion above, we get

$$\{\frac{h_j}{e_j} s_j + (\frac{P'_j}{e_j P_j} - \frac{h_j}{e_j}) e'_j\} V_j \ (JPY)$$

from the investment of the asset j. Then the value V'_x of the portfolio at the end of the period is represented as

$$V'_x = \sum_{j=1}^{n} \{\frac{h_j s_j}{e_j} + (\frac{P'_j}{e_j P_j} - \frac{h_j}{e_j}) e'_j\} V_j.$$

Let $r_j = (P'_j - P_j)/P_j$ for each $j \in \{1, \ldots, n\}$ be the return of the asset j, and let $r_x = (V'_x - V)/V$ be the return of the portfolio x. Using x_j and r_j, it is easy to see that

$$r_x = \sum_{j=1}^{n} \{\frac{h_j s_j}{e_j} + (1 - h_j + r_j) \frac{e'_j}{e_j} - 1\} x_j.$$

We consider that the return r_j and the exchange rate e'_j ($j \in \{1, 2, \ldots, n\}$) at the end of the investment period are random variables. Assume that we have T scenarios for these random variables and each scenario occurs with probability $1/T$. Let r_{jt} and e'_{jt} be the return and the exchange rate in the country $j \in \{1, 2, \ldots, n\}$ under the scenario $t \in \{1, 2, \ldots, T\}$. Then the return r_{xt} of the portfolio x under the scenario t is represented as

$$r_{xt} = \sum_{j=1}^{n} \{ \frac{h_j s_j}{e_j} + (1 - h_j + r_{jt}) \frac{e'_{jt}}{e_j} - 1 \} x_j.$$

The expected return of the portfolio x is written as

$$E[r_x] = \frac{1}{T} \sum_{t=1}^{T} r_{xt}.$$

We impose a constraint that the expected value of r_x is not less than a predetermined value r_E. The constraint is expressed as

$$E[r_x] \geq r_E.$$

We adopt CVaR as a risk measure of the portfolio. Let us define $l_x = -r_x$, namely the loss rate of the portfolio x. Under the scenario model (or the discrete probability model) mentioned above, CVaR of l_x with any probability level $\beta \in (0, 1)$ is defined as

$$CVaR_\beta[l_x] = \min_{\alpha} \left(\alpha + \frac{1}{(1-\beta)T} \sum_{t=1}^{T} \max\{l_{xt} - \alpha, 0\} \right)$$

by Rockafeller and Uryasev [1]. It is known that the minimum of the right hand side is attained when α is equal to the Value at Risk (VaR) with the probability level β.

By introducing variables $u_t = \max\{l_{xt} - \alpha, 0\} = \max\{-r_{xt} - \alpha, 0\}$ for each $t \in \{1, \ldots, T\}$, the model for finding the portfolio $x = (x_1, \ldots, x_n)$ and the vector $h = (h_1, \ldots, h_n)$ of hedge ratios, such that CVaR is minimized under the condition that the expected return of the portfolio is not less than r_E, is formulated as

$$\begin{array}{ll} \min & \alpha + \frac{1}{(1-\beta)T} \sum_{t=1}^{T} u_t \\ \text{s.t} & r_{xt} = \sum_{j=1}^{n} \{ \frac{h_j s_j}{e_j} + (1 - h_j + r_{jt}) \frac{e'_{jt}}{e_j} - 1 \} x_j \ (t = 1, \ldots, T) \\ & u_t \geq -r_{xt} - \alpha \ (t = 1, \ldots, T) \\ & u_t \geq 0 \ (t = 1, \ldots, T) \\ & \frac{1}{T} \sum_{t=1}^{T} r_{xt} \geq r_E \\ & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0 \ (j = 1, \ldots, n) \\ & 0 \leq h_j \leq 1 \ (j = 1, \ldots, n) \end{array}$$

or equivalently

$$\begin{array}{ll} \min & \alpha + \frac{1}{(1-\beta)T} \sum_{t=1}^{T} u_t \\ \text{s.t} & \sum_{j=1}^{n} \{ \frac{s_j}{e_j} h_j + (1 - h_j + r_{jt}) \frac{e'_{jt}}{e_j} - 1 \} x_j + \alpha + u_t \geq 0 \\ & u_t \geq 0 \ (t = 1, \ldots, T) \\ & \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{n} \{ \frac{s_j}{e_j} h_j + (1 - h_j + r_{jt}) \frac{e'_{jt}}{e_j} - 1 \} x_j \geq r_E \\ & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0 \ (j = 1, \ldots, n) \\ & 0 \leq h_j \leq 1 \ (j = 1, \ldots, n), \end{array} \tag{2.1}$$

where α, $(x_1, \ldots, x_n)$, $(h_1, \ldots, h_n)$, and $(u_1, \ldots, u_T)$ are variables. The model (2.1) is a nonlinear programming problem and it is not easy to solve. In the next theorem, we will show that an optimal solution of (2.1) is computed by solving a linear programming problem.

Theorem 2.1. *Let α^*, $(x_1^*, \ldots, x_n^*)$, $(z_1^*, \ldots, z_n^*)$, and $(u_1^*, \ldots, u_T^*)$ be optimal solutions for the next linear programming problem*

$$\begin{array}{ll} \min & \alpha + \frac{1}{(1-\beta)T} \sum_{t=1}^{T} u_t \\ \text{s.t} & \sum_{j=1}^{n} \{ (\frac{e'_{jt}}{e_j} + \frac{e'_{jt}}{e_j} r_{jt} - 1) x_j + \frac{s_j - e'_{jt}}{e_j} z_j \} + \alpha + u_t \geq 0 \\ & u_t \geq 0 \ (t = 1, \ldots, T) \\ & \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{n} \{ (\frac{e'_{jt}}{e_j} + \frac{e'_{jt}}{e_j} r_{jt} - 1) x_j + \frac{s_j - e'_{jt}}{e_j} z_j \} \geq r_E \\ & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0 \ (j = 1, \ldots, n) \\ & 0 \leq z_j \leq x_j \ (j = 1, \ldots, n). \end{array} \tag{2.2}$$

Define

$$h_j^* = \begin{cases} z_j^*/x_j^* & \text{if} \quad x_j^* > 0 \\ 0 & \text{if} \quad x_j^* = 0. \end{cases} \tag{2.3}$$

Then α^*, $(x_1^*, \ldots, x_n^*)$, $(h_1^*, \ldots, h_n^*)$, and $(u_1^*, \ldots, u_T^*)$ are optimal solutions for the problem (2.1).

Proof. We introduce new variables $z_j = x_j h_j$ for each $j \in \{1, \ldots, n\}$ in the problem (2.1). Since $x_j \geq 0$, the condition $0 \leq h_j \leq 1$ is equivalent to $0 \leq z_j \leq x_j$. By introducing these variables z_j, the problem (2.1) can be converted to the linear programming problem (2.2). Let α^*, $(x_1^*, \ldots, x_n^*)$, $(z_1^*, \ldots, z_n^*)$, and $(u_1^*, \ldots, u_T^*)$ be optimal solutions for the problem (2.2) and define $(h_1^*, \ldots, h_n^*)$ by (2.3). Then it is easy to see that α^*, $(x_1^*, \ldots, x_n^*)$, $(h_1^*, \ldots, h_n^*)$, and $(u_1^*, \ldots, u_T^*)$ are optimal solutions for the problem (2.1). □

3. Numerical experiments

For our experiments, we adopt five indices (or assets) in five countries as listed in Table 1. We assume that the length of one investment period is a month and we use rates of the monthly return of the assets. We collect data (rates of return, exchange rates, and interests rates) from June, 2011 to September, 2016, using online services. In this paper, we set $T = 40$, the number of scenarios. Our simulation starts from October, 2014. For this investment period, we use the prior 40 months data from June 2011 to September 2014 as scenarios and compute an optimal portfolio x^* and an optimal hedge vector h^* by solving the problem (2.2). After computing these vectors, we calculate the return of the portfolio when we hold it by one month. Then we shift the period to November, 2014 and compute an optimal portfolio and an optimal hedge vector by using data from July, 2011 to October, 2014, and so on. We repeat these process for 24 months until September, 2016. When we solve the problem (2.2), we set $\beta = 0.05$ and $r_E = 0.005$. We compare the proposed model with the no-hedge model and the full-hedge model. For the no-hedge model, the hedge ratios are set to zero, namely, $h_j = 0$ for each $j \in \{1, \ldots, n\}$ in the problem (2.1). On the other hand, for the full-hedge model, we set $h_j = 1$ for each $j \in \{1, 2, \ldots, n\}$.

The rates of the monthly return of portfolios obtained from the proposed model are described in Fig 1. We also show returns of no-hedge model, full-hedge model and TOPIX in the same figure. We can see from Fig 1 that the performance of the portfolio computed from the proposed model is more stable than the others. In Table 2, we show some results of

Table 1: List of five indices

Index	Country	Currency
TOPIX	Japan	JPY
S&P500	The United States	USD
DAX Index	Germany	EUR
FTSE100	Great Britain	GBP
SSE Composite Index	China	CNY

our experiments. As we can see from the table, the proposed model yields the best average return. On the other hand, CVaR and the standard deviation of the proposed model are not lowest. However, when we adjust return by these risk measures, the proposed model brings about the best result. Also, comparison with TOPIX clearly shows benefits of international diversified investment. From these observations, we can conclude that the proposed model is able to control the fluctuation of the returns in international diversified investment.

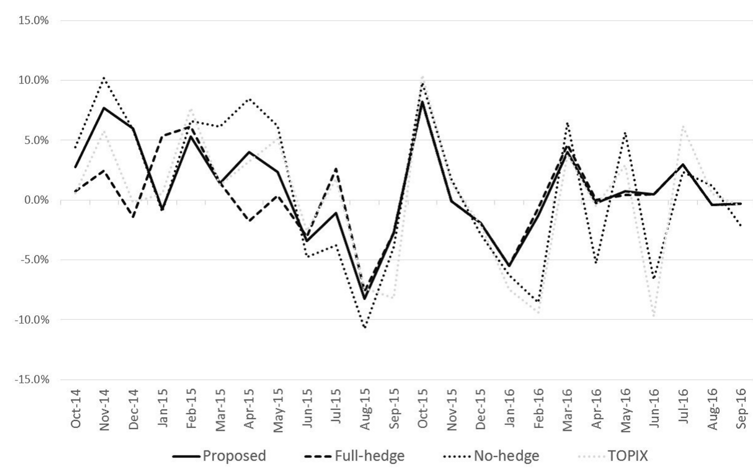

Fig 1: Monthly returns computed from three models and those of TOPIX

Table 2: Risk and return characteristics (in %)

	Optimal	Full-hedge	No-hedge	TOPIX
Average return of 24 months ①	0.82	0.43	0.80	0.13
CVaR of 24 months ②	6.87	6.55	9.64	9.54
①/②	0.120	0.066	0.083	0.014
Standard Deviation of 24 months③	3.85	3.47	6.05	5.31
①/③	0.214	0.124	0.132	0.025

In Fig 2 to Fig 4, we show how the portfolio of each model changes over the 24 months. From these figures, we see that the portfolio by each model drastically changes at September 2015. To investigate the differences, we show in Fig 5 currency exchange rates in the data period (from June 2011 to September 2016). We can see from Fig 5 that at the beginning of the period, JPY is relatively strong and the value of JPY keeps on decreasing until September 2015. After September 2015, the value of JPY again goes up. Recall that when we construct a portfolio at some point, we use data of the previous 40 months as scenarios.

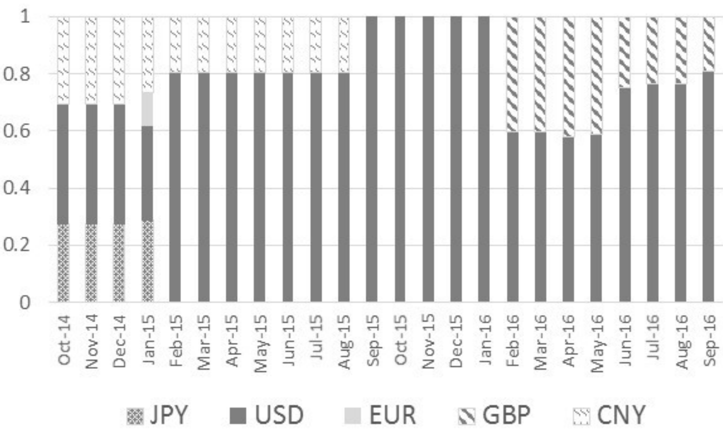

Fig 2: Portfolios by the proposed model

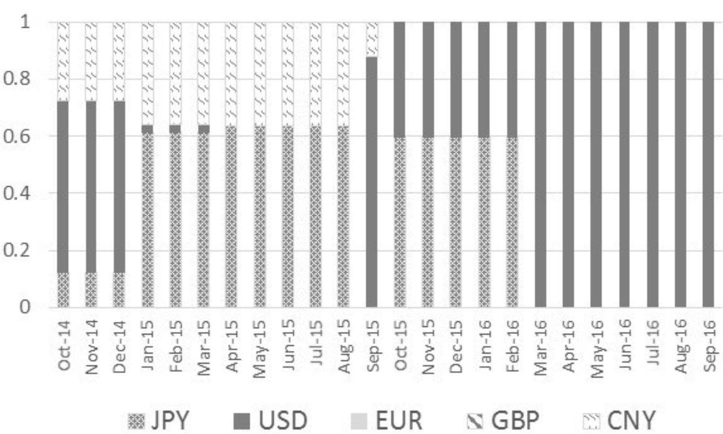

Fig 3: Portfolios by the no-hedge model

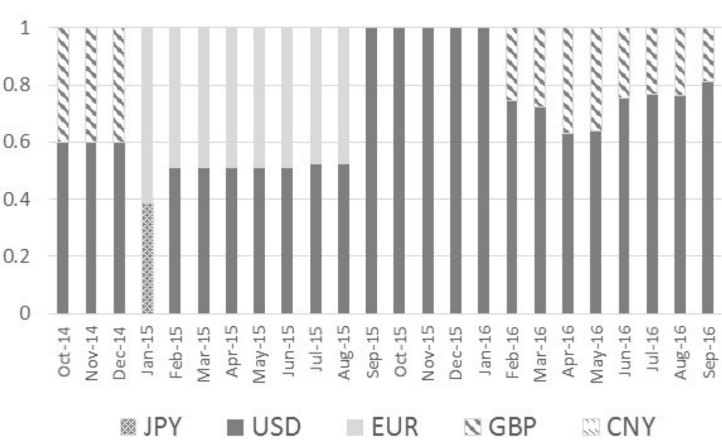

Fig 4: Portfolios by the full-hedge model

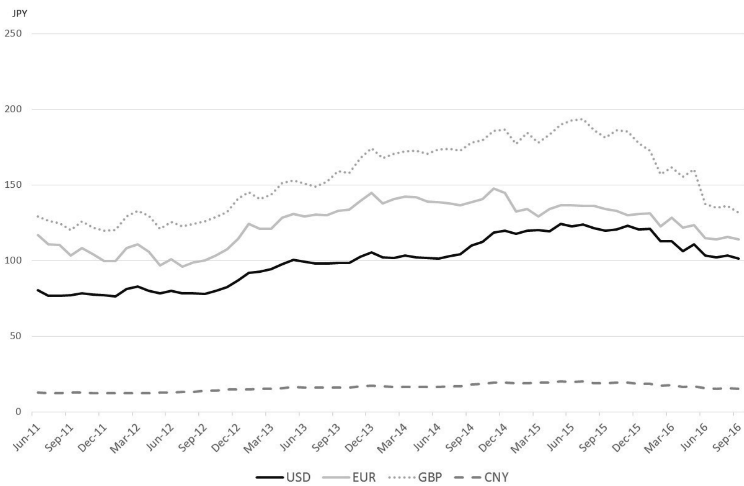

Fig 5: Currency exchange rates

If we take a point before September 2015, JPY is relatively strong for most of the 40 months. On the other hand, if we choose a point after September 2015, the 40 months' data include points when JPY is relatively weak. Intuitively speaking, when JPY is strong, we only need to hedge a little, so it is expected that the portfolio obtained by the proposed method is similar to that of the no-hedge model. On the other hand, when JPY is week, we have to hedge much, so we can expect the portfolio of the proposed model resembles the one by the full-hedge model.

To quantitatively assess these observations, we show average portfolios from October 2014 to September 2015 in Table 3, and those over the rest of the period in Table 4, respectively. It seems that Table 3 does not support our hypothesis. Then we look at Fig 1 again, and we see that the performance of the portfolio by the proposed method is more stable than that of the no-hedge model, and especially we can control the loss of the portfolio. Thus in this period, factors other than currency exchange rates affect the portfolios a lot. On the other hand, the hypothesis is supported in the last 12 month from Table 4.

Table 3: Average portfolios from October 2014 to September 2015 (in %)

	JPY	USD	EUR	GBP	CNY
Proposed model	10.08	65.81	1.05	0.00	23.06
No-hedge model	48.96	17.19	0.00	0.00	33.84
Full-hedge model	3.51	48.93	36.66	10.90	0.00

Table 4: Average portfolios from October 2015 to September 2016 (in %)

	JPY	USD	EUR	GBP	CNY
Proposed model	0.00	80.44	0.00	19.56	0.00
No-hedge model	22.90	76.16	0.00	0.00	0.94
Full-hedge model	0.00	83.33	0.00	16.67	0.00

To summarize, the proposed model automatically adjusts the portfolio and the hedge ratios according to market environments and we think this adjustment leads to the stable performance of the portfolio.

4. Conclusion

In this paper, we have developed a CVaR optimization model for international diversified investment. To reduce currency risks, we introduced forward contracts. The proposed model finds both a portfolio and hedge ratios of the forward contracts such that CVaR of the return is minimized under the condition that the expected return of a portfolio is not less than a predetermined value.

We conducted numerical experiments using actual market data. In the experiments, we compared the proposed model with no-hedge model and full-hedge model. As a result, we observed that the proposed model automatically adjusts the portfolio and the hedge ratios according to market environments, and this adjustment leads to the stable performance of the portfolio.

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