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Abstract $\mathrm{M}^\natural$-concave functions form a class of discrete concave functions in discrete convex analysis, and are defined by a certain exchange axiom. We show in this paper that $\mathrm{M}^\natural$-concave functions can be characterized by a combination of two simpler exchange properties. It is also shown that for a function defined on an integral polymatroid, a much simpler exchange axiom characterizes $\mathrm{M}^\natural$-concavity.

Keywords discrete convex analysis · discrete optimization · polymatroid · exchange property

Mathematics Subject Classification (2000) 90C27 · 52B40

1 Introduction and Results

Discrete convex analysis [14] is a theoretical framework for well-solved nonlinear discrete optimization problems, where a class of discrete concave functions, called $\mathrm{M}^\natural$-concave functions, plays a primary role. The concept of $\mathrm{M}^\natural$-concave function enjoys various nice mathematical properties, and efficient algorithms for discrete optimization problems with $\mathrm{M}^\natural$-concave objective functions have been proposed [6, 14].

$\mathrm{M}^\natural$-concave functions have also found applications in various research fields such as operations research, mathematical economics, and game theory (see, e.g., [7, 8, 10, 15, 17, 22]). An example is the resource allocation problem in operations research [10, 12, 20]. While the objective functions of the resource allocation problem are mostly assumed to be separable concave until mid 90s, $\mathrm{M}^\natural$-concavity provides a more general framework with nonseparable objective functions and efficient algorithms for the generalized problem.

The application of $\mathrm{M}^\natural$-concavity to mathematical economics was initiated by Danilov et al. [1, 2], where the existence of a Walrasian equilibrium in an exchange economy with indivisible goods is shown. The connection between discrete convex analysis and mathematical economics was accelerated by the observation of Fujishige–Yang [9] that $\mathrm{M}^\natural$-concavity for a set function is equivalent to the so-called "gross substitutes" property of Kelso–Crawford [11]. See [15] for more accounts on the application to mathematical economics.

In this paper, we discuss exchange axioms for $\mathrm{M}^\natural$-concave functions. Let n be a positive integer and $N = \{1, 2, \dots, n\}$. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be $\mathrm{M}^\natural$-concave if $\mathrm{dom}_{\mathbb{Z}} f$ is nonempty and f

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satisfies the following exchange property:

$$\textbf{(M}^\natural\textbf{-EXC}[\mathbb{Z}]\textbf{)}\ \forall x, y \in \mathbb{Z}^n,\ \forall i \in \mathrm{supp}^+(x-y):$$
$$f(x) + f(y) \le \max\Big[f(x-\chi_i) + f(y+\chi_i),\ \max_{j \in \mathrm{supp}^-(x-y)} \{f(x-\chi_i+\chi_j) + f(y+\chi_i-\chi_j)\}\Big],$$

where $\mathrm{dom}_{\mathbb{Z}} f = \{x \in \mathbb{Z}^n \mid f(x) > -\infty\}$ is the effective domain of f,

$$\mathrm{supp}^+(z) = \{i \in N \mid z(i) > 0\}, \quad \mathrm{supp}^-(z) = \{i \in N \mid z(i) < 0\} \quad (z \in \mathbb{Z}^n),$$

and for $i \in N$, $\chi_i \in \{0,1\}^n$ is the i-th unit vector, i.e., $\chi_i(i) = 1$, $\chi_i(j) = 0$ $(j \in N \setminus \{i\})$.

The main result of this paper, Theorem 1.1 below, says that M$^\natural$-concavity of functions defined on integer lattice points can be characterized by the combination of the following two exchange properties, where $x(N) = \sum_{i \in N} x(i)$:

$$\textbf{(P1}[\mathbb{Z}]\textbf{)}\ \forall x, y \in \mathbb{Z}^n \text{ with } x(N) < y(N):$$
$$f(x) + f(y) \le \max_{j \in \mathrm{supp}^-(x-y)} \{f(x+\chi_j) + f(y-\chi_j)\},$$
$$\textbf{(P2}[\mathbb{Z}]\textbf{)}\ \forall x, y \in \mathbb{Z}^n \text{ with } x(N) = y(N),\ \forall i \in \mathrm{supp}^+(x-y):$$
$$f(x) + f(y) \le \max_{j \in \mathrm{supp}^-(x-y)} \{f(x-\chi_i+\chi_j) + f(y+\chi_i-\chi_j)\}.$$

The first exchange property (P1[$\mathbb{Z}$]) applies to (x, y) with different component sums and ensures the possibility of making the pair closer with an appropriate unit vector χ_j. The second exchange property (P2[$\mathbb{Z}$]) applies to (x, y) with equal component sums and excludes the first possibility in (M$^\natural$-EXC[$\mathbb{Z}$]).

Theorem 1.1 *A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is M$^\natural$-concave if and only if* $\mathrm{dom}_{\mathbb{Z}} f \neq \emptyset$ *and f satisfies* (P1[$\mathbb{Z}$]) *and* (P2[$\mathbb{Z}$])*.*

Moreover, we show that the first property (P1[$\mathbb{Z}$]) alone characterizes M$^\natural$-concavity if the effective domain $\mathrm{dom}_{\mathbb{Z}} f$ is an M$^\natural$-convex set satisfying a certain condition; the definitions of an M$^\natural$-convex set and an integral polymatroid will be given in Section 2.

Theorem 1.2 *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function with* $\mathrm{dom}_{\mathbb{Z}} f \neq \emptyset$.
(i) *Suppose that* $\mathrm{dom}_{\mathbb{Z}} f$ *satisfies one of the following two conditions:*

(a) $\forall x, y \in \mathrm{dom}_{\mathbb{Z}} f,\ \exists z \in \mathrm{dom}_{\mathbb{Z}} f:\ z \le x,\ z \le y,$
(b) $\forall x, y \in \mathrm{dom}_{\mathbb{Z}} f,\ \exists z \in \mathrm{dom}_{\mathbb{Z}} f:\ z \ge x,\ z \ge y.$

Then, f is M$^\natural$-concave if and only if it satisfies (P1[$\mathbb{Z}$])*.*
(ii) *Suppose that* $\mathrm{dom}_{\mathbb{Z}} f$ *is an integral polymatroid. Then, f is M$^\natural$-concave if and only if it satisfies* (P1[$\mathbb{Z}$])*.*

M$^\natural$-concavity for set functions can be naturally defined through the one-to-one correspondence between set functions defined on 2^N and functions defined on $\{0,1\}^n$. A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is said to be M$^\natural$-concave if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies the following property[1]:

$$\textbf{(M}^\natural\textbf{-EXC}[\mathbb{B}]\textbf{)}\ \forall X, Y \subseteq N,\ \forall i \in X \setminus Y:$$
$$f(X) + f(Y) \le \max\Big[f(X-i) + f(Y+i),\ \max_{j \in Y \setminus X} \{f(X-i+j) + f(Y+i-j)\}\Big].$$

[1] $\mathbb{B}$ stands for Binary, referring to functions on $\{0,1\}^n$.

Here we use short-hand notations

$$X - i = X \setminus \{i\}, \qquad Y + i = Y \cup \{i\},$$
$$X - i + j = (X \setminus \{i\}) \cup \{j\}, \qquad Y + i - j = (Y \cup \{i\}) \setminus \{j\},$$

and denote $\mathrm{dom}\, f = \{X \subseteq N \mid f(X) > -\infty\}$.

For set functions, Theorems 1.1 and 1.2 can be specialized as follows, where the following exchange properties with cardinality restrictions are used:

(P1[$\mathbb{B}$]**)** $\forall X, Y \subseteq N$ with $|X| < |Y|$:

$$f(X) + f(Y) \le \max_{j \in Y \setminus X} \{f(X + j) + f(Y - j)\}$$

(P2[$\mathbb{B}$]**)** $\forall X, Y \subseteq N$ with $|X| = |Y|$, $\forall i \in X \setminus Y$:

$$f(X) + f(Y) \le \max_{j \in Y \setminus X} \{f(X - i + j) + f(Y + i - j)\}.$$

Corollary 1.3 *A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is M$^\natural$-concave if and only if* $\mathrm{dom}\, f \neq \emptyset$ *and f satisfies* (P1[$\mathbb{B}$]) *and* (P2[$\mathbb{B}$]).

Corollary 1.4 *Let $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ be a set function such that* $\mathrm{dom}\, f$ *contains the empty set. Then, f is M$^\natural$-concave if and only if it satisfies* (P1[$\mathbb{B}$]).

It is noted that if $\emptyset \in \mathrm{dom}\, f$ and f is M$^\natural$-concave, then $\mathrm{dom}\, f$ is necessarily the family of independent sets of a matroid.

2 Proof of Theorem 1.1

We prove Theorem 1.1 by showing Theorem 2.1 below that clarifies the relationship among various exchange axioms including (M$^\natural$-EXC[$\mathbb{Z}$]), (P1[$\mathbb{Z}$]), and (P2[$\mathbb{Z}$]).

By definition, an M$^\natural$-concave function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ satisfies the exchange property (M$^\natural$-EXC[$\mathbb{Z}$]). By setting $\chi_0 = \mathbf{0}$, the exchange property (M$^\natural$-EXC[$\mathbb{Z}$]) can be rewritten in a more compact form as

$$\forall x, y \in \mathrm{dom}_{\mathbb{Z}}\, f, \ \forall i \in \mathrm{supp}^+(x - y) :$$
$$f(x) + f(y) \le \max_{j \in \mathrm{supp}^-(x-y) \cup \{0\}} \{f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)\}.$$

It is easy to see that the effective domain $S = \mathrm{dom}_{\mathbb{Z}}\, f$ of an M$^\natural$-concave function satisfies the following property:

(B$^\natural$-EXC[$\mathbb{Z}$]**)** $\forall x, y \in S, \ \forall i \in \mathrm{supp}^+(x - y)$, (i) or (ii) (or both) holds:

(i) $x - \chi_i \in S, \ y + \chi_i \in S$,

(ii) $x - \chi_i + \chi_j \in S, \ y + \chi_i - \chi_j \in S$ for some $j \in \mathrm{supp}^-(x - y)$.

For a nonempty set $S \subseteq \mathbb{Z}^n$, we say that S is an M$^\natural$-convex set (an integral generalized polymatroid [4,5]) if it satisfies (B$^\natural$-EXC[$\mathbb{Z}$]). Hence, a nonempty set $S \subseteq \mathbb{Z}^n$ is an M$^\natural$-convex set if and only if its indicator function $\delta_S : \mathbb{Z}^n \to \{0, -\infty\}$ given by

$$\delta_S(x) = \begin{cases} 0 & (\text{if } x \in S), \\ -\infty & (\text{otherwise}) \end{cases}$$

is an M$^\natural$-concave function.

An example of an M$^\natural$-convex set is (the set of integral vectors in) an integral polymatroid. A nonempty set $P \subseteq \mathbb{Z}^n_+$ of nonnegative integral vectors is called an integral polymatroid if it satisfies the following condition (see, e.g., [23]):

$$\begin{array}{l} \text{(i) for } y \in P,\ x \in \mathbb{Z}^n_+,\ \text{if } x \leq y \text{ then } x \in P, \\ \text{(ii) if } x, y \in P,\ x(N) < y(N),\ \text{then } x + \chi_j \in P \text{ for some } j \in \mathrm{supp}^-(x - y). \end{array}$$

Every integral polymatroid is an M$^\natural$-convex set (see, e.g., [6,14,16]).

Theorem 2.1 below refers to the following additional exchange properties:

$$\begin{array}{l} \textbf{(P3}[\mathbb{Z}]\textbf{)}\ \forall x, y \in \mathrm{dom}_{\mathbb{Z}} f \text{ with } x(N) < y(N),\ \forall i \in \mathrm{supp}^+(x - y): \\ \qquad f(x) + f(y) \leq \displaystyle\max_{j \in \mathrm{supp}^-(x-y)} \{f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)\}, \\ \textbf{(P4}[\mathbb{Z}]\textbf{)}\ \forall x, y \in \mathrm{dom}_{\mathbb{Z}} f \text{ with } x(N) > y(N),\ \forall i \in \mathrm{supp}^+(x - y): \\ \qquad f(x) + f(y) \leq \displaystyle\max_{j \in \mathrm{supp}^-(x-y) \cup \{0\}} \{f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)\}, \end{array}$$

and the following local exchange properties:

$$\begin{array}{l} \textbf{(M}^\natural\textbf{-EXC}[\mathbb{Z}]_{\mathbf{loc}}\textbf{)} \\ \textbf{(L1}[\mathbb{Z}]\textbf{)}\ \forall x \in \mathbb{Z}^n,\ \forall i, j \in N: \quad f(x + \chi_i + \chi_j) + f(x) \leq f(x + \chi_i) + f(x + \chi_j), \\ \textbf{(L2}[\mathbb{Z}]\textbf{)}\ \forall x \in \mathbb{Z}^n,\ \forall i, j, k \in N \text{ with } k \notin \{i, j\}: \\ \quad f(x + \chi_i + \chi_j) + f(x + \chi_k) \\ \qquad\qquad \leq \max\left[f(x + \chi_i + \chi_k) + f(x + \chi_j),\ f(x + \chi_j + \chi_k) + f(x + \chi_i)\right], \\ \textbf{(L3}[\mathbb{Z}]\textbf{)}\ \forall x \in \mathbb{Z}^n,\ \forall i, j, k, l \in N,\ \{i, j\} \cap \{k, l\} = \emptyset: \\ f(x + \chi_i + \chi_j) + f(x + \chi_k + \chi_l) \\ \quad \leq \max\Big[f(x + \chi_i + \chi_k) + f(x + \chi_j + \chi_l),\ f(x + \chi_j + \chi_k) + f(x + \chi_i + \chi_l)\Big]. \end{array}$$

Here we allow the possibilities of $i = j$ in (L1[$\mathbb{Z}$]) and in (L2[$\mathbb{Z}$]), and $i = j$ or $k = l$ in (L3[$\mathbb{Z}$]).

Theorem 2.1 *For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}_{\mathbb{Z}} f \neq \emptyset$, the following conditions are equivalent:*

(i) f *satisfies* (M$^\natural$-EXC[$\mathbb{Z}$]),
(ii) f *satisfies* (P1[$\mathbb{Z}$]) *and* (P2[$\mathbb{Z}$]),
(iii) f *satisfies* (P2[$\mathbb{Z}$]), (P3[$\mathbb{Z}$]), *and* (P4[$\mathbb{Z}$]),
(iv) f *satisfies* (P1[$\mathbb{Z}$]), (P2[$\mathbb{Z}$]), (P3[$\mathbb{Z}$]), *and* (P4[$\mathbb{Z}$]),
(v) $\mathrm{dom}_{\mathbb{Z}} f$ *is an M$^\natural$-convex set and f satisfies* (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$).

In Theorem 2.1, the implications

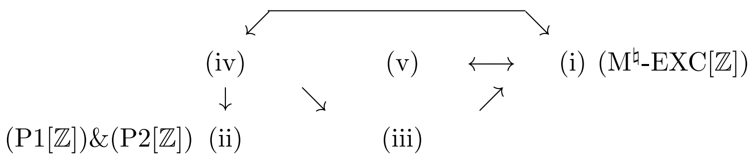
$$\begin{array}{ccccccc} & \text{(iv)} & & \text{(v)} & & \text{(i) (M}^\natural\text{-EXC}[\mathbb{Z}]) \\ & \downarrow & \searrow & & \nearrow & \\ (\text{P1}[\mathbb{Z}])\&(\text{P2}[\mathbb{Z}]) & \text{(ii)} & & \text{(iii)} & & \end{array}$$

are easy to see. We prove the implication "(i) $\Longrightarrow$ (v)" in Section 2.1 and "(v) $\Longrightarrow$ (iv)" in Section 2.2. The proof for the implication "(ii) $\Longrightarrow$ (iii)" is similar to that for "(v) $\Longrightarrow$ (iv)" and given in Section 2.3.

$$\begin{array}{ccccccc} & \text{(iv)} & \overset{\text{Sec. 2.2}}{\Longleftarrow} & \text{(v)} & \overset{\text{Sec. 2.1}}{\Longleftarrow} & \text{(i) (M}^\natural\text{-EXC}[\mathbb{Z}]) \\ & \downarrow & \searrow & & \nearrow & \\ (\text{P1}[\mathbb{Z}])\&(\text{P2}[\mathbb{Z}]) & \text{(ii)} & \overset{\text{Sec. 2.3}}{\Longrightarrow} & \text{(iii)} & & \end{array}$$

It should be clear that our primary interest lies in the equivalence of (i) and (ii) stated in Theorem 2.1, while the other conditions (iii) to (v) are introduced for the proof.

Remark 2.2 The equivalence of (i) and (iv) is already shown in [16] (see Theorem 4.1), and the equivalence of (i) and (v) has been known to experts (see Theorem 4.2). Thus the following implications are known:



□

2.1 Proof of "(i) $\Longrightarrow$ (v)" in Theorem 2.1

The condition (B$^\natural$-EXC[$\mathbb{Z}$]) for $\mathrm{dom}_{\mathbb{Z}} f$ is easy to see from the condition (M$^\natural$-EXC[$\mathbb{Z}$]). Hence, $\mathrm{dom}_{\mathbb{Z}} f$ is an M$^\natural$-convex set. The conditions (L1[$\mathbb{Z}$]) and (L2[$\mathbb{Z}$]) in (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$) are immediate consequences of (M$^\natural$-EXC[$\mathbb{Z}$]), whereas the third condition (L3[$\mathbb{Z}$]) is derived as follows.

We first consider the case where i, j, k, l are distinct. To simplify notations we assume $i = 1$, $j = 2$, $k = 3$, $l = 4$, and write $\alpha_1 = f(x + \chi_1)$, $\alpha_{23} = f(x + \chi_2 + \chi_3)$, $\alpha_{134} = f(x + \chi_1 + \chi_3 + \chi_4)$, and so on. Then the condition (L3[$\mathbb{Z}$]) can be rewritten as

$$\alpha_{12} + \alpha_{34} \le \max\{\alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\}.$$

We may assume $\alpha_{12} > -\infty$ and $\alpha_{34} > -\infty$ since otherwise this inequality is trivially true.

To prove the inequality by contradiction, suppose that

$$\alpha_{12} + \alpha_{34} > \max\{\alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\}. \tag{2.1}$$

With the notation $A = \alpha_{12} + \alpha_{34}$ we obtain

$$A = \alpha_{12} + \alpha_{34} \le \max\{\alpha_1 + \alpha_{234}, \alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\} = \alpha_1 + \alpha_{234} \tag{2.2}$$

from (M$^\natural$-EXC[$\mathbb{Z}$]) (with $i = 2$) and (2.1). Similarly, we have

$$A \le \alpha_2 + \alpha_{134}, \qquad A \le \alpha_3 + \alpha_{124}, \qquad A \le \alpha_4 + \alpha_{123}. \tag{2.3}$$

On the other hand, we have

$$\alpha_1 + \alpha_{123} \le \alpha_{12} + \alpha_{13}, \tag{2.4}$$

$$\alpha_2 + \alpha_{234} \le \alpha_{23} + \alpha_{24}, \tag{2.5}$$

$$\alpha_3 + \alpha_{134} \le \alpha_{13} + \alpha_{34}, \tag{2.6}$$

$$\alpha_4 + \alpha_{124} \le \alpha_{14} + \alpha_{24} \tag{2.7}$$

by (M$^\natural$-EXC[$\mathbb{Z}$]). By adding the four inequalities in (2.2) and (2.3) and using the inequalities in (2.4)–(2.7) and (2.1), we obtain

$$\begin{aligned} 4A &\le (\alpha_1 + \alpha_{234}) + (\alpha_2 + \alpha_{134}) + (\alpha_3 + \alpha_{124}) + (\alpha_4 + \alpha_{123}) \\ &= (\alpha_1 + \alpha_{123}) + (\alpha_2 + \alpha_{234}) + (\alpha_3 + \alpha_{134}) + (\alpha_4 + \alpha_{124}) \\ &\le (\alpha_{12} + \alpha_{13}) + (\alpha_{23} + \alpha_{24}) + (\alpha_{13} + \alpha_{34}) + (\alpha_{14} + \alpha_{24}) \\ &= (\alpha_{12} + \alpha_{34}) + (\alpha_{23} + \alpha_{14}) + 2(\alpha_{13} + \alpha_{24}) \\ &< 4A. \end{aligned}$$

This is a contradiction, and hence (L3[$\mathbb{Z}$]) is shown for distinct i, j, k, l.

When $i = j$, the above proof still works with the understanding that $\alpha_{12} = f(x+\chi_1+\chi_2) = f(x+2\chi_1)$ and $\alpha_{123} = f(x + 2\chi_1 + \chi_3)$, etc. Similarly in the case of $k = l$.

2.2 Proof of "(v) $\Longrightarrow$ (iv)" in Theorem 2.1

We assume that $\mathrm{dom}_{\mathbb{Z}} f$ satisfies (B$^\natural$-EXC[$\mathbb{Z}$]) and f satisfies (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$), and show that the four conditions (P1[$\mathbb{Z}$]), (P2[$\mathbb{Z}$]), (P3[$\mathbb{Z}$]), and (P4[$\mathbb{Z}$]) hold.

We first present two lemmas on the properties of a set $S \subseteq \mathbb{Z}^n$ satisfying (B$^\natural$-EXC[$\mathbb{Z}$]).

Lemma 2.3 *If a nonempty set $S \subseteq \mathbb{Z}^n$ satisfies* (B$^\natural$-EXC[$\mathbb{Z}$]), *then for any $x, y \in S$ with $x(N) < y(N)$ there exists some $j \in \mathrm{supp}^-(x-y)$ such that $y - \chi_j \in S$.*

Proof We prove the claim by induction on $\|x-y\|_1$. If $\|x-y\|_1 = 1$, then we have $x = y - \chi_j \in S$ for the unique $j \in \mathrm{supp}^-(x-y)$. Hence, the claim holds.

For the induction step, we assume $\|x-y\|_1 > 1$. If $\mathrm{supp}^+(x-y) = \emptyset$, then (B$^\natural$-EXC[$\mathbb{Z}$]) applied to y, x, and an arbitrarily chosen $j \in \mathrm{supp}^+(y-x)$ implies $y - \chi_j \in S$. Hence, we assume $\mathrm{supp}^+(x-y) \neq \emptyset$, and take any $i \in \mathrm{supp}^+(x-y)$. Then, (B$^\natural$-EXC[$\mathbb{Z}$]) applied to x, y, and i implies that there exists some $k \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that $x - \chi_i + \chi_k \in S$. Putting $x' = x - \chi_i + \chi_k$, we have $x'(N) \leq x(N) < y(N)$ and $\|x'-y\|_1 < \|x-y\|_1$. Hence, we can apply the induction hypothesis to x' and y to obtain $y - \chi_j \in S$ for some $j \in \mathrm{supp}^-(x'-y) \subseteq \mathrm{supp}^-(x-y)$. This concludes the proof. □

Lemma 2.4 *If a nonempty set $S \subseteq \mathbb{Z}^n$ satisfies* (B$^\natural$-EXC[$\mathbb{Z}$]), *then for any $x, y \in S$ with $x(N) \leq y(N)$ and $i \in \mathrm{supp}^+(x-y)$, there exists $j \in \mathrm{supp}^-(x-y)$ such that $y + \chi_i - \chi_j \in S$.*

Proof By (B$^\natural$-EXC[$\mathbb{Z}$]) applied to x, y, and $i \in \mathrm{supp}^+(x-y)$, we have $y + \chi_i - \chi_j \in S$ for some $j \in \mathrm{supp}^-(x-y) \cup \{0\}$. If $j \neq 0$, then we are done. Suppose that $j = 0$ holds. Since $x(N) \leq y(N) < (y+\chi_i)(N)$, we can apply Lemma 2.3 to obtain $y + \chi_i - \chi_j \in S$ for some $j \in \mathrm{supp}^-(x-(y+\chi_i)) = \mathrm{supp}^-(x-y)$. □

In the following, we prove the four conditions (P1[$\mathbb{Z}$]), (P2[$\mathbb{Z}$]), (P3[$\mathbb{Z}$]), and (P4[$\mathbb{Z}$]) in turn. We first prove (P1[$\mathbb{Z}$]).

Lemma 2.5 *For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$, if $\mathrm{dom}_{\mathbb{Z}} f$ satisfies* (B$^\natural$-EXC[$\mathbb{Z}$]) *and f satisfies* (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$), *then* (P1[$\mathbb{Z}$]) *holds.*

Proof To prove (P1[$\mathbb{Z}$]) by contradiction, we assume that the set of pairs

$$\begin{aligned}\mathcal{D} = \{(x,y) \mid\ & x, y \in \mathrm{dom}_{\mathbb{Z}} f,\ x(N) < y(N),\\ & f(x) + f(y) > f(x+\chi_j) + f(y-\chi_j)\ (\forall j \in \mathrm{supp}^-(x-y))\}\end{aligned}$$

is nonempty. Take a pair $(x,y) \in \mathcal{D}$ with $\|x-y\|_1$ minimum. For a fixed $\varepsilon > 0$, consider a vector $p \in \mathbb{R}^n$ with each component $p(j)$ $(j \in N)$ given by

$$p(j) = \begin{cases} f(x) - f(x+\chi_j) & (\text{if } j \in \mathrm{supp}^-(x-y),\ x+\chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \\ f(y-\chi_j) - f(y) + \varepsilon & (\text{if } j \in \mathrm{supp}^-(x-y),\ x+\chi_j \notin \mathrm{dom}_{\mathbb{Z}} f, \\ & \qquad y-\chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \\ 0 & (\text{otherwise}). \end{cases}$$

We use notation $f_p(x) = f(x) + \langle p, x\rangle$ for $x \in \mathbb{Z}^n$.

Claim 1:

$$f_p(x+\chi_j) = f_p(x) \qquad (\forall j \in \mathrm{supp}^-(x-y) \text{ with } x+\chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \tag{2.8}$$

$$f_p(y-\chi_j) < f_p(y) \qquad (\forall j \in \mathrm{supp}^-(x-y)). \tag{2.9}$$

(Proof of Claim 1) The equality (2.8) is obvious from the definition of p. If $x + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$, then the inequality (2.9) follows from (2.8) and

$$f_p(x) + f_p(y) > f_p(x+\chi_j) + f_p(y-\chi_j).$$

If $x+\chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, (2.9) follows from the fact that $f_p(y-\chi_j)-f_p(y) = -\varepsilon$ or $-\infty$ depending on whether $y-\chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ or not.

In the following, we consider the two cases and derive a contradiction in each case.

$$\text{(Case 1): } y(N)-x(N) \geq 2, \qquad \text{(Case 2): } y(N)-x(N) = 1.$$

We first consider Case 1. Since $x(N) < y(N)$, Lemma 2.3 implies $y-\chi_{j_0} \in \mathrm{dom}_{\mathbb{Z}} f$ for some $j_0 \in \mathrm{supp}^-(x-y)$.

Claim 2: $(x,y') \in \mathcal{D}$ for $y' = y-\chi_{j_0}$.
(Proof of Claim 2) Since $y'(N) = y(N)-1 > x(N)$, we only have to show that

$$f_p(x)+f_p(y') > f_p(x+\chi_j)+f_p(y'-\chi_j) \quad (\forall j \in \mathrm{supp}^-(x-y')). \tag{2.10}$$

Since this inequality is obvious when $x+\chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, we assume that $x+\chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Then we have $f_p(x) = f_p(x+\chi_j)$ by (2.8). We also have

$$\begin{aligned} f_p(y'-\chi_j) &= [f_p(y)+f_p(y-\chi_{j_0}-\chi_j)] - f_p(y) \\ &\leq [f_p(y-\chi_{j_0})+f_p(y-\chi_j)] - f_p(y) \\ &< f_p(y-\chi_{j_0}) = f_p(y') \end{aligned}$$

by (M$^\natural$-EXC$[\mathbb{Z}]_{\mathrm{loc}}$) ((L1$[\mathbb{Z}]$), in particular) and (2.9). Therefore, (2.10) holds. This establishes Claim 2.

Claim 2 contradicts the choice of (x,y) since $\|x-y'\|_1 = \|x-y\|_1 - 1$. Therefore, Case 1 cannot occur. We next consider Case 2.

Claim 3: There exist $i_0 \in \mathrm{supp}^+(x-y)$ and $j_0 \in \mathrm{supp}^-(x-y)$ such that $y+\chi_{i_0}-\chi_{j_0} \in \mathrm{dom}_{\mathbb{Z}} f$ and

$$f_p(y+\chi_{i_0}-\chi_{j_0}) \geq f_p(y+\chi_{i_0}-\chi_j) \quad (\forall j \in \mathrm{supp}^-(x-y)). \tag{2.11}$$

(Proof of Claim 3) We first note that $\mathrm{supp}^+(x-y) \neq \emptyset$. Indeed, if $\mathrm{supp}^+(x-y) = \emptyset$, then we have $x = y-\chi_i$ and $y = x+\chi_i$ for the unique element $i \in \mathrm{supp}^-(x-y)$, and $f(x)+f(y) = f(x+\chi_i)+f(y-\chi_i)$, a contradiction to $(x,y) \in \mathcal{D}$.

Take some $i_0 \in \mathrm{supp}^+(x-y)$. Lemma 2.4 implies $y+\chi_{i_0}-\chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ for some $j \in \mathrm{supp}^-(x-y)$. Any $j \in \mathrm{supp}^-(x-y)$ that maximizes $f_p(y+\chi_{i_0}-\chi_j)$ serves as j_0.

Claim 4: $(x,y') \in \mathcal{D}$ for $y' = y+\chi_{i_0}-\chi_{j_0}$.
(Proof of Claim 4) Since $y'(N) = y(N) > x(N)$, we only have to show that

$$f_p(x)+f_p(y') > f_p(x+\chi_j)+f_p(y'-\chi_j) \quad (\forall j \in \mathrm{supp}^-(x-y')). \tag{2.12}$$

Since this inequality is obvious when $x+\chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, we assume that $x+\chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Then we have $f_p(x) = f_p(x+\chi_j)$ by (2.8). We also have

$$\begin{aligned} & f_p(y'-\chi_j) \\ & = [f_p(y)+f_p(y+\chi_{i_0}-\chi_{j_0}-\chi_j)] - f_p(y) \\ & \leq \max\{f_p(y+\chi_{i_0}-\chi_{j_0})+f_p(y-\chi_j), \\ & \qquad\qquad f_p(y+\chi_{i_0}-\chi_j)+f_p(y-\chi_{j_0})\} - f_p(y) \\ & \leq f_p(y+\chi_{i_0}-\chi_{j_0}) + \max\{f_p(y-\chi_j)-f_p(y),\ f_p(y-\chi_{j_0})-f_p(y)\} \\ & < f_p(y+\chi_{i_0}-\chi_{j_0}) = f_p(y') \end{aligned}$$

by (M$^\natural$-EXC$[\mathbb{Z}]_{\mathrm{loc}}$) ((L2$[\mathbb{Z}]$), in particular), (2.11), and (2.9). Therefore, (2.12) holds. This establishes Claim 4.

Claim 4 contradicts the choice of (x,y) since $\|x-y'\|_1 = \|x-y\|_1 - 2$. Therefore, Case 2 cannot occur either. Hence, $\mathcal{D}$ must be empty, which means that (P1$[\mathbb{Z}]$) is true. This concludes the proof of the lemma. □

We next prove (P2[$\mathbb{Z}$]) and (P3[$\mathbb{Z}$]) simultaneously.

Lemma 2.6 *For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$, if $\mathrm{dom}_{\mathbb{Z}} f$ satisfies* (B$^\natural$-EXC[$\mathbb{Z}$]) *and f satisfies* (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$)*, then* (P2[$\mathbb{Z}$]) *and* (P3[$\mathbb{Z}$]) *hold; that is, for any $x, y \in \mathbb{Z}^n$ with $x(N) \le y(N)$ and $i \in \mathrm{supp}^+(x-y)$ we have*

$$f(x) + f(y) \le \max_{j \in \mathrm{supp}^-(x-y)} \{f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)\}. \tag{2.13}$$

Proof To prove (2.13) by contradiction, we assume that there exists a pair (x, y) for which (2.13) fails. That is, we assume that the set of pairs

$$\begin{aligned}\mathcal{D} = \{(x,y) \mid x, y \in \mathrm{dom}_{\mathbb{Z}} f,\ x(N) \le y(N),\ \exists i_* \in \mathrm{supp}^+(x-y) \text{ s.t.}\\ f(x) + f(y) > f(x - \chi_{i_*} + \chi_j) + f(y + \chi_{i_*} - \chi_j)\\ (\forall j \in \mathrm{supp}^-(x-y))\}\end{aligned}$$

is nonempty. Take a pair $(x, y) \in \mathcal{D}$ with $\|x - y\|_1$ minimum, and fix $i_* \in \mathrm{supp}^+(x-y)$ appearing in the definition of $\mathcal{D}$. We have $\|x-y\|_1 \ge 3$ by (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$). For a fixed $\varepsilon > 0$, consider a vector $p \in \mathbb{R}^n$ with each component $p(j)$ $(j \in N)$ given by

$$p(j) = \begin{cases} f(x) - f(x - \chi_{i_*} + \chi_j) & (\text{if } j \in \mathrm{supp}^-(x-y),\ x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \\ f(y + \chi_{i_*} - \chi_j) - f(y) + \varepsilon & (\text{if } j \in \mathrm{supp}^-(x-y),\ x - \chi_{i_*} + \chi_j \not\in \mathrm{dom}_{\mathbb{Z}} f, \\ & \qquad\qquad y + \chi_{i_*} - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \\ 0 & (\text{otherwise}). \end{cases}$$

Recall the notation $f_p(x) = f(x) + \langle p, x \rangle$ for $x \in \mathbb{Z}^n$.

Claim 1:

$$f_p(x - \chi_{i_*} + \chi_j) = f_p(x) \qquad (\forall j \in \mathrm{supp}^-(x-y) \text{ with } x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \tag{2.14}$$

$$f_p(y + \chi_{i_*} - \chi_j) < f_p(y) \quad (\forall j \in \mathrm{supp}^-(x-y)). \tag{2.15}$$

(Proof of Claim 1) The equality (2.14) is obvious from the definition of p. If $x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$, then (2.15) follows from (2.14) and

$$f_p(x) + f_p(y) > f_p(x - \chi_{i_*} + \chi_j) + f_p(y + \chi_{i_*} - \chi_j).$$

If $x - \chi_{i_*} + \chi_j \not\in \mathrm{dom}_{\mathbb{Z}} f$, (2.15) follows from the fact that $f_p(y + \chi_{i_*} - \chi_j) - f_p(y) = -\varepsilon$ or $-\infty$ depending on whether $y + \chi_{i_*} - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ or not.

In the following, we consider the following two cases and derive a contradiction in each case.

(Case 1): $x(N) < y(N)$, (Case 2): $x(N) = y(N)$.

We first consider Case 1.

Claim 2: There exists $j_0 \in \mathrm{supp}^-(x-y)$ such that $y - \chi_{j_0} \in \mathrm{dom}_{\mathbb{Z}} f$ and

$$f_p(y - \chi_{j_0}) \ge f_p(y - \chi_j) \qquad (\forall j \in \mathrm{supp}^-(x-y)). \tag{2.16}$$

(Proof of Claim 2) Since $x(N) < y(N)$, Lemma 2.3 implies $y - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ for some $j \in \mathrm{supp}^-(x-y)$. Any $j \in \mathrm{supp}^-(x-y)$ that maximizes $f_p(y - \chi_j)$ serves as j_0.

Claim 3: $(x, y') \in \mathcal{D}$ for $y' = y - \chi_{j_0}$.

(Proof of Claim 3) First note that $x(N) \leq y(N) - 1 = y'(N)$. We are to show

$$f_p(x) + f_p(y') > f_p(x - \chi_{i_*} + \chi_j) + f_p(y' + \chi_{i_*} - \chi_j) \quad (\forall j \in \mathrm{supp}^-(x - y')). \tag{2.17}$$

Since this inequality is obvious when $x - \chi_{i_*} + \chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, we assume that $x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Then we have $f_p(x) = f_p(x - \chi_{i_*} + \chi_j)$ by (2.14). We also have

$$\begin{aligned}
& f_p(y' + \chi_{i_*} - \chi_j) \\
&= [f_p(y + \chi_{i_*} - \chi_{j_0} - \chi_j) + f_p(y)] - f_p(y) \\
&\leq \max\{f_p(y - \chi_{j_0}) + f_p(y + \chi_{i_*} - \chi_j), \\
&\qquad\qquad f_p(y - \chi_j) + f_p(y + \chi_{i_*} - \chi_{j_0})\} - f_p(y) \\
&\leq f_p(y - \chi_{j_0}) + \max\{f_p(y + \chi_{i_*} - \chi_j) - f_p(y),\ f_p(y + \chi_{i_*} - \chi_{j_0}) - f_p(y)\} \\
&< f_p(y - \chi_{j_0}) = f_p(y'),
\end{aligned}$$

where the first inequality is due to (M$^\natural$-EXC$[\mathbb{Z}]_{\mathrm{loc}}$) ((L2$[\mathbb{Z}]$), in particular), and the second and third inequalities are by (2.16) and (2.15). Hence follows (2.17), establishing Claim 3.

Claim 3 contradicts the choice of (x, y) since $\|x - y'\|_1 = \|x - y\|_1 - 1$. Therefore, Case 1 cannot occur. We next consider Case 2. Note that $\|x - y\|_1 \geq 4$ in this case.

Claim 4: There exist $i_0 \in \mathrm{supp}^+(x - y)$ and $j_0 \in \mathrm{supp}^-(x - y)$ such that $y + \chi_{i_0} - \chi_{j_0} \in \mathrm{dom}_{\mathbb{Z}} f$ and

$$f_p(y + \chi_{i_0} - \chi_{j_0}) \geq f_p(y + \chi_{i_0} - \chi_j) \qquad (\forall j \in \mathrm{supp}^-(x - y)). \tag{2.18}$$

In addition, we can take such i_0 with $i_0 \neq i_*$ if $x(i_*) = y(i_*) + 1$.
(Proof of Claim 4) Let i_0 be any element in $\mathrm{supp}^+(x - y)$; if $x(i_*) = y(i_*) + 1$, then we may assume $i_0 \neq i_*$ since $\sum_{i \in \mathrm{supp}^+(x-y)}(x(i) - y(i)) \geq 2$ holds. By Lemma 2.4, there exists $j \in \mathrm{supp}^-(x - y)$ such that $y + \chi_{i_0} - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Any $j \in \mathrm{supp}^-(x - y)$ that maximizes $f_p(y + \chi_{i_0} - \chi_j)$ serves as j_0.

Claim 5: $(x, y') \in \mathcal{D}$ for $y' = y + \chi_{i_0} - \chi_{j_0}$.
(Proof of Claim 5) First note that $x(N) = y(N) = y'(N)$ and $i_* \in \mathrm{supp}^+(x - y')$ by the choice of y'. We are to show

$$f_p(x) + f_p(y') > f_p(x - \chi_{i_*} + \chi_j) + f_p(y' + \chi_{i_*} - \chi_j) \ (\forall j \in \mathrm{supp}^-(x - y')). \tag{2.19}$$

Since this inequality is obvious when $x - \chi_{i_*} + \chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, we assume that $x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Then we have $f_p(x) = f_p(x - \chi_{i_*} + \chi_j)$ by (2.14). We also have

$$\begin{aligned}
& f_p(y' + \chi_{i_*} - \chi_j) \\
&= [f_p(y + \chi_{i_0} + \chi_{i_*} - \chi_{j_0} - \chi_j) + f_p(y)] - f_p(y) \\
&\leq \max\{f_p(y + \chi_{i_0} - \chi_{j_0}) + f_p(y + \chi_{i_*} - \chi_j), \\
&\qquad\qquad f_p(y + \chi_{i_0} - \chi_j) + f_p(y + \chi_{i_*} - \chi_{j_0})\} - f_p(y) \\
&\leq f_p(y + \chi_{i_0} - \chi_{j_0}) \\
&\qquad\qquad + \max\{f_p(y + \chi_{i_*} - \chi_j) - f_p(y),\ f_p(y + \chi_{i_*} - \chi_{j_0}) - f_p(y)\} \\
&< f_p(y + \chi_{i_0} - \chi_{j_0}) = f_p(y'),
\end{aligned}$$

where the first inequality is due to (M$^\natural$-EXC$[\mathbb{Z}]_{\mathrm{loc}}$) ((L3$[\mathbb{Z}]$), in particular), and the second and third inequalities are by (2.18) and (2.15). Hence follows (2.19), establishing Claim 5.

Claim 5 contradicts the choice of (x, y), since $\|x - y'\|_1 = \|x - y\|_1 - 2$. Therefore, Case 2 cannot occur, either. Hence, $\mathcal{D}$ must be empty, which means that (2.13) is true. This concludes the proof of the lemma □

We finally prove (P4$[\mathbb{Z}]$).

Lemma 2.7 *For a function* $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$*, if* $\mathrm{dom}_{\mathbb{Z}} f$ *satisfies* (B$^\natural$-EXC[$\mathbb{Z}$]) *and* f *satisfies* (M$^\natural$-EXC[$\mathbb{Z}$]$_{\rm loc}$)*, then* f *satisfies* (P4[$\mathbb{Z}$]).

Proof To prove (P4[$\mathbb{Z}$]) by contradiction, we assume that the set of pairs

$$\begin{aligned}\mathcal{D} = \{(x,y) \mid x, y \in \mathrm{dom}_{\mathbb{Z}} f,\ x(N) > y(N),\ \exists i_* \in \mathrm{supp}^+(x-y) \text{ s.t.}\\ f(x)+f(y) > f(x-\chi_{i_*}+\chi_j) + f(y+\chi_{i_*}-\chi_j)\\ (\forall j \in \mathrm{supp}^-(x-y)\cup\{0\})\}\end{aligned}$$

is nonempty; recall that $\chi_0 = (0,0,\ldots,0)$. We note that $\|x-y\|_1 \geq 2$ for every $(x,y) \in \mathcal{D}$. Indeed, if $\|x-y\|_1 = 1$, then we have $x = y + \chi_i$ and $y = x - \chi_i$ with the unique element $i \in \mathrm{supp}^+(x-y)$, and therefore $f(x)+f(y) = f(x-\chi_i)+f(y+\chi_i)$ holds, implying that $(x,y) \notin \mathcal{D}$. Take a pair $(x,y) \in \mathcal{D}$ with $\|x-y\|_1$ minimum, and fix $i_* \in \mathrm{supp}^+(x-y)$ appearing in the definition of $\mathcal{D}$.

For a fixed $\varepsilon > 0$, define $p \in \mathbb{R}^n$ as follows. The component $p(i_*)$ is defined by

$$p(i_*) = \begin{cases} f(x-\chi_{i_*}) - f(x) & (\text{if } x-\chi_{i_*} \in \mathrm{dom}_{\mathbb{Z}} f),\\ f(y) - f(y+\chi_{i_*}) - \varepsilon & (\text{if } x-\chi_{i_*} \notin \mathrm{dom}_{\mathbb{Z}} f,\ y+\chi_{i_*} \in \mathrm{dom}_{\mathbb{Z}} f),\\ 0 & (\text{if } x-\chi_{i_*} \notin \mathrm{dom}_{\mathbb{Z}} f,\ y+\chi_{i_*} \notin \mathrm{dom}_{\mathbb{Z}} f).\end{cases}$$

The component $p(j)$ for each $j \in \mathrm{supp}^-(x-y)$ is defined by

$$p(j) = \begin{cases} f(x) - f(x-\chi_{i_*}+\chi_j) + p(i_*) & (\text{if } x-\chi_{i_*}+\chi_j \in \mathrm{dom}_{\mathbb{Z}} f),\\ f(y+\chi_{i_*}-\chi_j) - f(y) + p(i_*) + \varepsilon & \\ \qquad (\text{if } x-\chi_{i_*}+\chi_j \notin \mathrm{dom}_{\mathbb{Z}} f,\ y+\chi_{i_*}-\chi_j \in \mathrm{dom}_{\mathbb{Z}} f),\\ 0 \qquad (\text{if } x-\chi_{i_*}+\chi_j \notin \mathrm{dom}_{\mathbb{Z}} f,\ y+\chi_{i_*}-\chi_j \notin \mathrm{dom}_{\mathbb{Z}} f).\end{cases}$$

We set $p(j) = 0$ for all other components of p. Recall the notation $f_p(x) = f(x) + \langle p, x\rangle$ for $x \in \mathbb{Z}^n$.

Claim 1:

$$f_p(x-\chi_{i_*}+\chi_j) = f_p(x) \quad (\forall j \in \mathrm{supp}^-(x-y)\cup\{0\} \text{ with } x-\chi_{i_*}+\chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \tag{2.20}$$

$$f_p(y+\chi_{i_*}-\chi_j) < f_p(y) \qquad (\forall j \in \mathrm{supp}^-(x-y)\cup\{0\}). \tag{2.21}$$

(Proof of Claim 1) The equality (2.20) is obvious from the definition of p. If $x - \chi_{i_*} + j \in \mathrm{dom}_{\mathbb{Z}} f$, then the inequality (2.21) follows from (2.20) and

$$f_p(x) + f_p(y) > f_p(x-\chi_{i_*}+\chi_j) + f_p(y+\chi_{i_*}-\chi_j);$$

otherwise, (2.21) follows from the fact that $f_p(y+\chi_{i_*}-\chi_j) - f_p(y) = -\varepsilon$ or $-\infty$ depending on whether $y+\chi_{i_*}-\chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ or not.

Claim 2: There exist $i_0 \in \mathrm{supp}^+(x-y)\setminus\{i_*\}$ and $j_0 \in \mathrm{supp}^-(x-y)\cup\{0\}$ such that $y+\chi_{i_0}-\chi_{j_0} \in \mathrm{dom}_{\mathbb{Z}} f$ and

$$f_p(y+\chi_{i_0}-\chi_{j_0}) \geq f_p(y+\chi_{i_0}-\chi_j) \qquad (\forall j \in \mathrm{supp}^-(x-y)\cup\{0\}). \tag{2.22}$$

(Proof of Claim 2) Since $x(N) > y(N)$, Lemma 2.5 implies the inequality

$$f(x) + f(y) \leq f(x-\chi_{i_0}) + f(y+\chi_{i_0})$$

for some $i_0 \in \mathrm{supp}^+(x-y)$, from which follows that $x-\chi_{i_0} = x-\chi_{i_0}+\chi_0 \in \mathrm{dom}_{\mathbb{Z}} f$. Since $(x,y) \in \mathcal{D}$, we have $i_0 \neq i_*$, i.e., $i_0 \in \mathrm{supp}^+(x-y)\setminus\{i_*\}$. Any $j \in \mathrm{supp}^-(x-y)\cup\{0\}$ that maximizes $f_p(y+\chi_{i_0}-\chi_j)$ serves as j_0.

Claim 3: For $y' = y + \chi_{i_0} - \chi_{j_0}$ we have

$$f_p(x) + f_p(y') > f_p(x - \chi_{i_*} + \chi_j) + f_p(y' + \chi_{i_*} - \chi_j) \qquad (\forall j \in \mathrm{supp}^-(x - y') \cup \{0\}). \quad (2.23)$$

(Proof of Claim 3) First note that $i_* \in \mathrm{supp}^+(x-y')$ by the choice of y'. Since this inequality is obvious when $x-\chi_{i_*}+\chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, we assume that $x-\chi_{i_*}+\chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Then we have $f_p(x) = f_p(x-\chi_{i_*}+\chi_j)$ by (2.20). We also have

$$\begin{aligned}
& f_p(y' + \chi_{i_*} - \chi_j) \\
& = [f_p(y + \chi_{i_0} + \chi_{i_*} - \chi_{j_0} - \chi_j) + f_p(y)] - f_p(y) \\
& \leq \max\{f_p(y + \chi_{i_0} - \chi_{j_0}) + f_p(y + \chi_{i_*} - \chi_j), \\
& \qquad\qquad f_p(y + \chi_{i_0} - \chi_j) + f_p(y + \chi_{i_*} - \chi_{j_0})\} - f_p(y) \\
& \leq f_p(y + \chi_{i_0} - \chi_{j_0}) \\
& \qquad\qquad + \max\{f_p(y + \chi_{i_*} - \chi_j) - f_p(y),\ f_p(y + \chi_{i_*} - \chi_{j_0}) - f_p(y)\} \\
& < f_p(y + \chi_{i_0} - \chi_{j_0}) = f_p(y'),
\end{aligned}$$

where the first, second, and third inequalities follow from ($\mathrm{M}^\natural$-EXC$[\mathbb{Z}]_{\mathrm{loc}}$), (2.22), and (2.21), respectively. Therefore, (2.23) holds. This completes the proof of Claim 3.

By $x(N) > y(N)$ and $y'(N) \leq y(N) + 1$ we have $x(N) \geq y'(N)$, in which the possibility of equality is excluded. Indeed, if $x(N) = y'(N)$, then (2.23) contradicts (2.13) in Lemma 2.6 for (x, y', i_*). Therefore, $x(N) > y'(N)$ holds. Hence, we have $(x, y') \in \mathcal{D}$ by Claim 3, a contradiction to the choice of (x, y), since $\|x - y'\|_1 \leq \|x - y\|_1 - 1$. Therefore, $\mathcal{D}$ must be empty, which means that (P4$[\mathbb{Z}]$) is satisfied. □

2.3 Proof of "(ii) $\Longrightarrow$ (iii)" in Theorem 2.1

We prove the implication "(ii) $\Longrightarrow$ (iii)" in Theorem 2.1 by showing that the conditions (P1$[\mathbb{Z}]$) and (P2$[\mathbb{Z}]$) imply (P3$[\mathbb{Z}]$) and (P4$[\mathbb{Z}]$).

We first deal with the condition (P3$[\mathbb{Z}]$).

Lemma 2.8 *If $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ satisfies* (P1$[\mathbb{Z}]$) *and* (P2$[\mathbb{Z}]$)*, then it also satisfies* (P3$[\mathbb{Z}]$).

Proof To prove (P3$[\mathbb{Z}]$) by contradiction, we assume that the set of pairs

$$\begin{aligned}
\mathcal{D} = \{(x, y) \mid x, y \in \mathrm{dom}_{\mathbb{Z}} f,\ x(N) < y(N),\ \exists i_* \in \mathrm{supp}^+(x - y) \text{ s.t.} \\
f(x) + f(y) > f(x - \chi_{i_*} + \chi_j) + f(y + \chi_{i_*} - \chi_j) \\
(\forall j \in \mathrm{supp}^-(x - y))\}
\end{aligned}$$

is nonempty. Take a pair $(x, y) \in \mathcal{D}$ with $\|x - y\|_1$ minimum, and fix $i_* \in \mathrm{supp}^+(x - y)$ appearing in the definition of $\mathcal{D}$.

For a fixed $\varepsilon > 0$, consider a vector $p \in \mathbb{R}^n$ with each component $p(j)$ $(j \in N)$ given by

$$p(j) = \begin{cases} f(x) - f(x - \chi_{i_*} + j) & (\text{if } j \in \mathrm{supp}^-(x - y),\ x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \\ f(y + \chi_{i_*} - j) - f(y) + \varepsilon & (\text{if } j \in \mathrm{supp}^-(x - y),\ x - \chi_{i_*} + \chi_j \notin \mathrm{dom}_{\mathbb{Z}} f, \\ & \qquad y + \chi_{i_*} - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \\ 0 & (\text{otherwise}). \end{cases}$$

Recall the notation $f_p(x) = f(x) + \langle p, x \rangle$ for $x \in \mathbb{Z}^n$.

Claim 1:

$$f_p(x - \chi_{i_*} + \chi_j) = f_p(x) \qquad (\forall j \in \mathrm{supp}^-(x-y) \text{ with } x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \tag{2.24}$$

$$f_p(y + \chi_{i_*} - \chi_j) < f_p(y) \qquad (\forall j \in \mathrm{supp}^-(x-y)). \tag{2.25}$$

(Proof of Claim 1) The equality (2.24) is obvious from the definition of p. If $x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$, then the inequality (2.25) follows from (2.24) and

$$f_p(x) + f_p(y) > f_p(x - \chi_{i_*} + \chi_j) + f_p(y + \chi_{i_*} - \chi_j).$$

If $x - \chi_{i_*} + \chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, (2.25) follows from the fact that $f_p(y + \chi_{i_*} - \chi_j) - f_p(y) = -\varepsilon$ or $-\infty$ depending on whether $y + \chi_{i_*} - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ or not.

Claim 2: There exists $j_0 \in \mathrm{supp}^-(x-y)$ such that $y - \chi_{j_0} \in \mathrm{dom}_{\mathbb{Z}} f$ and

$$f_p(y - \chi_{j_0}) \geq f_p(y - \chi_j) \qquad (\forall j \in \mathrm{supp}^-(x-y)). \tag{2.26}$$

(Proof of Claim 2) Since $x(N) < y(N)$, the condition (P1[$\mathbb{Z}$]) implies that there exists $j \in \mathrm{supp}^-(x-y)$ such that $y - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Any $j \in \mathrm{supp}^-(x-y)$ that maximizes $f_p(y - \chi_j)$ serves as j_0.

Claim 3: For $y' = y - \chi_{j_0}$ we have

$$f_p(x) + f_p(y') > f_p(x - \chi_{i_*} + \chi_j) + f_p(y' + \chi_{i_*} - \chi_j) \qquad (\forall j \in \mathrm{supp}^-(x-y')). \tag{2.27}$$

(Proof of Claim 3) Since this inequality is obvious when $x - \chi_{i_*} + \chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, we assume that $x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Then we have $f_p(x) = f_p(x - \chi_{i_*} + \chi_j)$ by (2.24). We also have

$$\begin{aligned}
& f_p(y' + \chi_{i_*} - \chi_j) \\
& = [f_p(y + \chi_{i_*} - \chi_{j_0} - \chi_j) + f_p(y)] - f_p(y) \\
& \leq \max\{f_p(y - \chi_{j_0}) + f_p(y + \chi_{i_*} - \chi_j), \\
& \qquad\qquad f_p(y - \chi_j) + f_p(y + \chi_{i_*} - \chi_{j_0})\} - f_p(y) \\
& \leq f_p(y - \chi_{j_0}) + \max\{f_p(y + \chi_{i_*} - \chi_j) - f_p(y),\ f_p(y + \chi_{i_*} - \chi_{j_0}) - f_p(y)\} \\
& < f_p(y - \chi_{j_0}) = f_p(y'),
\end{aligned}$$

where the first inequality is due to (P1[$\mathbb{Z}$]), and the second and third inequalities are by (2.26) and (2.25). Hence follows (2.27), establishing Claim 3.

By $x(N) < y(N)$ and $y'(N) = y(N) - 1$ we have $x(N) \leq y'(N)$, in which the possibility of equality is excluded. Indeed, if $x(N) = y'(N)$, then (2.27) contradicts (P2[$\mathbb{Z}$]) for (x, y', i_*). Therefore, $x(N) < y'(N)$ holds. Hence, we have $(x, y') \in \mathcal{D}$ by Claim 3, a contradiction to the choice of (x, y) since $\|x - y'\|_1 = \|x - y\|_1 - 1$. Therefore, $\mathcal{D}$ must be empty, which means that (P3[$\mathbb{Z}$]) is true. □

We next show that (P4[$\mathbb{Z}$]) holds.

Lemma 2.9 *If $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ satisfies* (P1[$\mathbb{Z}$]) *and* (P2[$\mathbb{Z}$])*, then it also satisfies* (P4[$\mathbb{Z}$])*.*

Proof To prove (P4[$\mathbb{Z}$]) by contradiction, we assume, to the contrary, that the set of pairs

$$\begin{aligned}
\mathcal{D} = \{(x, y) \mid\ & x, y \in \mathrm{dom}_{\mathbb{Z}} f,\ x(N) > y(N),\ \exists i_* \in \mathrm{supp}^+(x-y) \text{ s.t.} \\
& f(x) + f(y) > f(x - \chi_{i_*} + \chi_j) + f(y + \chi_{i_*} - \chi_j) \\
& \qquad (\forall j \in \mathrm{supp}^-(x-y) \cup \{0\})\}
\end{aligned}$$

is nonempty; recall that $\chi_0 = (0, 0, \ldots, 0)$. Take a pair $(x, y) \in \mathcal{D}$ with $\|x - y\|_1$ minimum, and fix $i_* \in \mathrm{supp}^+(x - y)$ appearing in the definition of $\mathcal{D}$.

For a fixed $\varepsilon > 0$, define $p \in \mathbb{R}^n$ as follows. The component $p(i_*)$ is defined by

$$p(i_*) = \begin{cases} f(x - \chi_{i_*}) - f(x) & (\text{if } x - \chi_{i_*} \in \mathrm{dom}_{\mathbb{Z}} f), \\ f(y) - f(y + \chi_{i_*}) - \varepsilon & (\text{if } x - \chi_{i_*} \notin \mathrm{dom}_{\mathbb{Z}} f,\ y + \chi_{i_*} \in \mathrm{dom}_{\mathbb{Z}} f), \\ 0 & (\text{otherwise}). \end{cases}$$

The component $p(j)$ for each $j \in \mathrm{supp}^-(x - y)$ is defined by

$$p(j) = \begin{cases} f(x) - f(x - \chi_{i_*} + \chi_j) + p(i_*) & (\text{if } x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \\ f(y + \chi_{i_*} - \chi_j) - f(y) + p(i_*) + \varepsilon & \\ \qquad (\text{if } x - \chi_{i_*} + \chi_j \notin \mathrm{dom}_{\mathbb{Z}} f,\ y + \chi_{i_*} - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), & \\ 0 & (\text{otherwise}). \end{cases}$$

We set $p(j) = 0$ for all other components of p. Recall the notation $f_p(x) = f(x) + \langle p, x \rangle$ for $x \in \mathbb{Z}^n$.

Claim 1:

$$f_p(x - \chi_{i_*} + \chi_j) = f_p(x) \qquad (\forall j \in \mathrm{supp}^-(x - y) \cup \{0\} \text{ with } x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f), \tag{2.28}$$

$$f_p(y + \chi_{i_*} - \chi_j) < f_p(y) \qquad (\forall j \in \mathrm{supp}^-(x - y) \cup \{0\}). \tag{2.29}$$

(Proof of Claim 1) The equality (2.28) is obvious from the definition of p. If $x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$, then the inequality (2.29) follows from (2.28) and

$$f_p(x) + f_p(y) > f_p(x - \chi_{i_*} + \chi_j) + f_p(y + \chi_{i_*} - \chi_j);$$

otherwise, (2.29) follows from the fact that $f_p(y + \chi_{i_*} - \chi_j) - f_p(y) = -\varepsilon$ or $-\infty$ depending on whether $y + \chi_{i_*} - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ or not.

Claim 2: There exist $i_0 \in \mathrm{supp}^+(x - y) \setminus \{i_*\}$ and $j_0 \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that $y + \chi_{i_0} - \chi_{j_0} \in \mathrm{dom}_{\mathbb{Z}} f$ and

$$f_p(y + \chi_{i_0} - \chi_{j_0}) \geq f_p(y + \chi_{i_0} - \chi_j) \qquad (\forall j \in \mathrm{supp}^-(x - y) \cup \{0\}). \tag{2.30}$$

(Proof of Claim 2) Since $x(N) > y(N)$, the condition (P1[$\mathbb{Z}$]) applied to x and y implies the inequality

$$f(x) + f(y) \leq f(x - \chi_{i_0}) + f(y + \chi_{i_0})$$

for some $i_0 \in \mathrm{supp}^+(x - y)$, where we have $i_0 \neq i_*$ since $(x, y) \in \mathcal{D}$. This inequality implies $y + \chi_{i_0} - \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ with $j = 0$. Any $j \in \mathrm{supp}^-(x - y) \cup \{0\}$ that maximizes $f_p(y + \chi_{i_0} - \chi_j)$ serves as j_0.

Claim 3: For $y' = y + \chi_{i_0} - \chi_{j_0}$ we have

$$f_p(x) + f_p(y') > f_p(x - \chi_{i_*} + \chi_j) + f_p(y' + \chi_{i_*} - \chi_j) \qquad (\forall j \in \mathrm{supp}^-(x - y') \cup \{0\}). \tag{2.31}$$

(Proof of Claim 3) First note that $i_* \in \mathrm{supp}^+(x - y')$ by the choice of y'. Since the inequality (2.31) is obvious when $x - \chi_{i_*} + \chi_j \notin \mathrm{dom}_{\mathbb{Z}} f$, we assume that $x - \chi_{i_*} + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$. Then we have $f_p(x) = f_p(x - \chi_{i_*} + \chi_j)$ by (2.28). We also have

$$\begin{aligned} & f_p(y' + \chi_{i_*} - \chi_j) \\ & = [f_p(y + \chi_{i_0} + \chi_{i_*} - \chi_{j_0} - \chi_j) + f_p(y)] - f_p(y) \\ & \leq \max\{f_p(y + \chi_{i_0} - \chi_{j_0}) + f_p(y + \chi_{i_*} - \chi_j), \\ & \qquad\qquad f_p(y + \chi_{i_0} - \chi_j) + f_p(y + \chi_{i_*} - \chi_{j_0})\} - f_p(y) \\ & \leq f_p(y + \chi_{i_0} - \chi_{j_0}) + \max\{f_p(y + \chi_{i_*} - \chi_j) - f_p(y),\ f_p(y + \chi_{i_*} - \chi_{j_0}) - f_p(y)\} \\ & < f_p(y + \chi_{i_0} - \chi_{j_0}) = f_p(y'), \end{aligned}$$

where the first inequality is by (P1[$\mathbb{Z}$]) or (P2[$\mathbb{Z}$]), and the second and third inequalities follow from (2.30) and (2.29), respectively. Therefore, (2.31) holds. This completes the proof of Claim 3.

By $x(N) > y(N)$ and $y'(N) \le y(N) + 1$ we have $x(N) \ge y'(N)$, in which the possibility of equality is excluded. Indeed, if $x(N) = y'(N)$, then (2.31) contradicts (P2[$\mathbb{Z}$]) for (x, y', i_*). Therefore, $x(N) > y'(N)$ holds. Hence, we have $(x, y') \in \mathcal{D}$ by Claim 3, a contradiction to the choice of (x, y), since $\|x - y'\|_1 \le \|x - y\|_1 - 1$. Therefore, $\mathcal{D}$ must be empty, which means that the condition (P4[$\mathbb{Z}$]) is satisfied. □

3 Proof of Theorem 1.2

The polymatroid case in claim (ii) of Theorem 1.2 follows immediately from the general case (i) since the condition (a) in (i) holds for an integral polymatroid; recall that an integral polymatroid is a set of nonnegative vectors containing the zero vector $\mathbf{0}$. Hence, it suffices to prove the claim (i) of Theorem 1.2.

Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function such that $\mathrm{dom}_{\mathbb{Z}} f$ is a nonempty set satisfying the condition (a) or (b); we may assume that (a) holds since the other case can be proven in a similar way (alternatively, we apply the case (a) to $f(-x)$). The implication "(M$^\natural$-EXC[$\mathbb{Z}$]) $\Longrightarrow$ (P1[$\mathbb{Z}$])" is already shown in Theorem 2.1. To prove the converse, we show below that the property (P1[$\mathbb{Z}$]) implies M$^\natural$-convexity for $\mathrm{dom}_{\mathbb{Z}} f$ and (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$) for f. Then, f is an M$^\natural$-concave function by Theorem 2.1.

[Proof of M$^\natural$-convexity for $\mathrm{dom}_{\mathbb{Z}} f$] We use the fact that an M$^\natural$-convex set can be characterized by the following exchange properties.

- $\forall x, y \in S,\ x(N) < y(N),\ \exists j \in \mathrm{supp}^-(x - y) : x + \chi_j,\ y - \chi_j \in S,$ (3.1)
- $\forall x, y \in S,\ x(N) = y(N),\ \forall i \in \mathrm{supp}^+(x - y),$
  $\exists j \in \mathrm{supp}^-(x - y) : x - \chi_i + \chi_j,\ y + \chi_i - \chi_j \in S,$ (3.2)
- $\forall x, y \in S,\ x(N) = y(N),\ \forall i \in \mathrm{supp}^+(x - y),$
  $\exists j \in \mathrm{supp}^-(x - y) : x - \chi_i + \chi_j \in S.$ (3.3)

Proposition 3.1 *For a nonempty set $S \subseteq \mathbb{Z}^n$, the following conditions are equivalent:*
(i) *S is an M$^\natural$-convex set (i.e., satisfies* (B$^\natural$-EXC[$\mathbb{Z}$])*),*
(ii) *S satisfies* (3.1) *and* (3.2)*.*
(iii) *S satisfies* (3.1) *and* (3.3)*.*

Proof The equivalence between (i) and (ii) follows from Theorem 2.1 applied to the indicator function δ_S. It is well known (see, e.g., [14, Prop. 4.2]) that a set S satisfies the condition (3.2) if and only if it satisfies (3.3). □

We also use the following property of a set satisfying (3.1).

Proposition 3.2 *If $S \subseteq \mathbb{Z}^n$ satisfies* (3.1)*, then for every $x, y \in S$ with $x \le y$, it holds that $[x, y] \subseteq S$, where $[x, y] = \{z \in \mathbb{Z}^n \mid x \le z \le y\}$.*

Proof We prove the claim by induction on $\|x - y\|_1$. If $\|x - y\|_1 = 0$ then the claim holds. Assume $\|x - y\|_1 > 0$. Then, we have $x(N) < y(N)$, and (3.1) shows the existence of $j \in \mathrm{supp}^-(x - y)$ such that $x' = x + \chi_j \in S$ and $y' = y - \chi_j \in S$. Since $\|x' - y\|_1 < \|x - y\|_1$ and $\|x - y'\|_1 < \|x - y\|_1$ hold, the induction hypothesis implies that $[x, y] \subseteq [x', y] \cup [x, y'] \subseteq S$. □

By Proposition 3.1, the effective domain $\mathrm{dom}_{\mathbb{Z}} f$ is an M$^\natural$-convex set if (and only if) it satisfies (3.1) and (3.3). The condition (3.1) for $\mathrm{dom}_{\mathbb{Z}} f$ follows immediately from (P1[$\mathbb{Z}$]) for f. It follows from (3.1) and Proposition 3.2 that

$$x, y \in \mathrm{dom}_{\mathbb{Z}} f,\ x \le y \Longrightarrow [x, y] \subseteq \mathrm{dom}_{\mathbb{Z}} f. \quad (3.4)$$

We now prove (3.3) for $\mathrm{dom}_{\mathbb{Z}} f$. Let $x, y \in \mathrm{dom}_{\mathbb{Z}} f$ be vectors with $x(N) = y(N)$ and $i \in \mathrm{supp}^+(x-y)$. By the condition (a), there exists some $z \in \mathrm{dom}_{\mathbb{Z}} f$ such that $z \leq x$ and $z \leq y$. Since $x(i) > y(i) \geq z(i)$, we have $x' = x - \chi_i \in [z, x] \subseteq \mathrm{dom}_{\mathbb{Z}} f$ by (3.4). Since $x'(N) < y(N)$ and $x', y \in \mathrm{dom}_{\mathbb{Z}} f$, (P1[$\mathbb{Z}$]) implies that $x' + \chi_j = x - \chi_i + \chi_j \in \mathrm{dom}_{\mathbb{Z}} f$ for some $j \in \mathrm{supp}^-(x' - y) \subseteq \mathrm{supp}^-(x - y)$. Thus, the condition (3.3) holds for $\mathrm{dom}_{\mathbb{Z}} f$.

[Proof of (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$) for f] The properties (L1[$\mathbb{Z}$]) and (L2[$\mathbb{Z}$]) are immediate consequences of (P1[$\mathbb{Z}$]). The condition (L3[$\mathbb{Z}$]) is derived as follows. We first consider the case where i, j, k, l are distinct. To simplify the notation we assume that $i = 1$, $j = 2$, $k = 3$, $l = 4$, and write $\alpha_1 = f(x + \chi_1)$, $\alpha_{23} = f(x + \chi_2 + \chi_3)$, and so on. Then, the condition (L3[$\mathbb{Z}$]) can be rewritten as

$$\alpha_{12} + \alpha_{34} \leq \max\{\alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\}. \tag{3.5}$$

We may assume $\alpha_{12} > -\infty$ and $\alpha_{34} > -\infty$ since otherwise the inequality (3.5) is trivially true. This means that $x + \chi_1 + \chi_2 \in \mathrm{dom}_{\mathbb{Z}} f$ and $x + \chi_3 + \chi_4 \in \mathrm{dom}_{\mathbb{Z}} f$. Then, we have some $z \in \mathrm{dom}_{\mathbb{Z}} f$ with $z \leq x$ by the condition (a), and therefore (3.4) implies that for each $i \in \{1, 2, 3, 4\}$, we have $x + \chi_i \in \mathrm{dom}_{\mathbb{Z}} f$, i.e., $\alpha_i > -\infty$.

To prove the inequality (3.5), assume, to the contrary, that

$$\alpha_{12} + \alpha_{34} > \max\{\alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\}. \tag{3.6}$$

By the property (L2[$\mathbb{Z}$]) applied to $x + \chi_1 + \chi_2$ and $x + \chi_3$, we may assume, by symmetry of 1 and 2, that

$$\alpha_{12} + \alpha_3 \leq \alpha_{13} + \alpha_2. \tag{3.7}$$

By (3.6) and (3.7), we have

$$\alpha_{34} + \alpha_2 > \alpha_{24} + \alpha_3.$$

On the other hand, by the condition (L2[$\mathbb{Z}$]), we have

$$\alpha_{34} + \alpha_2 \leq \max\{\alpha_{23} + \alpha_4,\ \alpha_{24} + \alpha_3\}. \tag{3.8}$$

Hence

$$\alpha_{34} + \alpha_2 \leq \alpha_{23} + \alpha_4. \tag{3.9}$$

Similarly, it follows from (L2[$\mathbb{Z}$]) that

$$\alpha_{12} + \alpha_4 \leq \max\{\alpha_{14} + \alpha_2,\ \alpha_{24} + \alpha_1\}, \tag{3.10}$$

$$\alpha_{34} + \alpha_1 \leq \max\{\alpha_{13} + \alpha_4,\ \alpha_{14} + \alpha_3\}. \tag{3.11}$$

In (3.10) we have

$$\alpha_{12} + \alpha_4 \leq \alpha_{24} + \alpha_1 \tag{3.12}$$

from (3.6) and (3.9). In (3.11) we have

$$\alpha_{34} + \alpha_1 \leq \alpha_{14} + \alpha_3. \tag{3.13}$$

from (3.6) and (3.12). By adding the inequalities (3.7), (3.9), (3.12), and (3.13) we obtain

$$\begin{aligned}
&2(\alpha_{12} + \alpha_{34}) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \\
&\leq (\alpha_{13} + \alpha_{24}) + (\alpha_{14} + \alpha_{23}) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \\
&\leq 2\max\{\alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\} + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4),
\end{aligned}$$

a contradiction to (3.6), and hence (L3[$\mathbb{Z}$]) is shown for distinct i, j, k, l.

When $i = j$, the above proof still works with the understanding that $\alpha_{12} = f(x+\chi_1+\chi_2) = f(x+2\chi_1)$ and $\alpha_{23} = f(x + \chi_1 + \chi_3)$, etc. Similarly in the case of $k = l$. Thus, (L3[$\mathbb{Z}$]), and hence (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$) hold for f.

This concludes the proof of the claim (i) in Theorem 2.1.

4 Concluding Remarks

4.1 Connection with M-concave Functions

The concept of M$^\natural$-concave function is originally introduced as a variant of M-concave function. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be M-concave [14] if it satisfies the following exchange property:

$$\textbf{(M-EXC}[\mathbb{Z}]\textbf{)}\ \forall x, y \in \mathbb{Z}^n,\ \forall i \in \mathrm{supp}^+(x-y) :$$
$$f(x) + f(y) \leq \max_{j \in \mathrm{supp}^-(x-y)} \{f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)\}.$$

In its original definition, an M$^\natural$-concave function is a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ such that the function $\tilde{f} : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ defined from f as

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & (\text{if } x_0 = -x(N)), \\ -\infty & (\text{otherwise (i.e., } x_0 \neq -x(N))) \end{cases} \quad (x_0 \in \mathbb{Z},\ x \in \mathbb{Z}^n) \tag{4.1}$$

is an M-concave function.

It is easy to see that the condition (M-EXC[$\mathbb{Z}$]) for $\tilde{f}$ is equivalent to the combination of the properties (P1[$\mathbb{Z}$]), (P2[$\mathbb{Z}$]), (P3[$\mathbb{Z}$]), and (P4[$\mathbb{Z}$]) for f (i.e., the condition (iv) in Theorem 2.1). The following theorem [16] shows the equivalence of these conditions and their equivalence to (M$^\natural$-EXC[$\mathbb{Z}$]).

Theorem 4.1 ([16, Th. 4.2]) *For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}_\mathbb{Z} f \neq \emptyset$, define a function $\tilde{f} : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ by* (4.1)*. Then,*

$$\begin{aligned} \tilde{f} \text{ is M-concave} \quad &\Longleftrightarrow \quad f \text{ satisfies (P1}[\mathbb{Z}]\text{), (P2}[\mathbb{Z}]\text{), (P3}[\mathbb{Z}]\text{), and (P4}[\mathbb{Z}]\text{)} \\ &\Longleftrightarrow \quad f \text{ satisfies (M}^\natural\text{-EXC}[\mathbb{Z}]\text{).} \end{aligned}$$

It is known that the effective domain $\mathrm{dom}_\mathbb{Z} f$ of an M-concave function is contained in a hyperplane of the form $\{x \in \mathbb{Z}^n \mid x(N) = \alpha\}$ for some $\alpha \in \mathbb{Z}$. Hence, for an M-concave function, the condition (P1[$\mathbb{Z}$]) is void, and (P2[$\mathbb{Z}$]) follows immediately from (M-EXC[$\mathbb{Z}$]), which, together with Theorem 1.1, implies the known fact that the class of M-concave functions is (properly) contained in the class of M$^\natural$-concave functions (see [14]).

On the other hand, Theorem 1.1 shows that the restriction of an M$^\natural$-concave function on a hyperplane of the form $\{x \in \mathbb{Z}^n \mid x(N) = \alpha\}$ is an M-concave function [16, Th. 3.1].

4.2 Characterization by Local Exchange Properties

Theorem 2.1 contains the following characterization of M$^\natural$-concave function by the local exchange property (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$), which deserves to be stated as a separate theorem.

Theorem 4.2 *A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}_\mathbb{Z} f \neq \emptyset$ satisfies* (M$^\natural$-EXC[$\mathbb{Z}$]) *if and only if $\mathrm{dom}_\mathbb{Z} f$ is an M$^\natural$-convex set and f satisfies* (M$^\natural$-EXC[$\mathbb{Z}$]$_{\mathrm{loc}}$)*.*

This theorem has been known to experts, though not stated explicitly, as a corollary of the following characterization of M-concave functions by a local exchange property. An M-convex set means an M$^\natural$-convex set contained in a hyperplane of the form $\{x \in \mathbb{Z}^n \mid x(N) = \alpha\}$.

Theorem 4.3 ([13, Th. 3.1]) *A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is M-concave if and only if $\mathrm{dom}_\mathbb{Z} f$ is an M-convex set and f satisfies the property* (L3[$\mathbb{Z}$])*.*

Proof (Proof of Theorem 4.2) For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$, we consider the function $\tilde{f} : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given by (4.1). By Theorems 4.1 and 4.3, f satisfies (M$^\natural$-EXC[$\mathbb{Z}$]) if and only if $\mathrm{dom}_{\mathbb{Z}}\, \tilde{f}$ is an M-convex set and $\tilde{f}$ satisfies (L3[$\mathbb{Z}$]). Note that the effective domain $\mathrm{dom}_{\mathbb{Z}}\, \tilde{f}$ is contained in the hyperplane $x_0 + x(N) = 0$. It is easy to see that the property (L3[$\mathbb{Z}$]) for $\tilde{f}$ can be rewritten in terms of f as the property (M$^\natural$-EXC$[\mathbb{Z}]_{\mathrm{loc}}$), and it is known that $\mathrm{dom}_{\mathbb{Z}}\, \tilde{f}$ is an M-convex set if and only if $\mathrm{dom}_{\mathbb{Z}}\, f$ is an M$^\natural$-convex set [14, Sec. 4.7]. □

From Theorem 1.1 and the proof of Theorem 1.2 we see that the combination of local exchange properties (L1[$\mathbb{Z}$]) and (L2[$\mathbb{Z}$]) characterizes M$^\natural$-concavity under a certain assumption of the effective domain. This is stated in the following theorem, where the first part (i) is deeply related to [21, Th. 6.5 (i)] and the second part (ii) generalizes the results in [19, Th. 10] and in [21, Th. 6.5 (ii)] for the case with $\mathrm{dom}_{\mathbb{Z}}\, f \subseteq \{0, 1\}^n$.

Theorem 4.4 *Let* $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ *be a function with* $\mathrm{dom}_{\mathbb{Z}}\, f \neq \emptyset$.
(i) *Suppose that* $\mathrm{dom}_{\mathbb{Z}}\, f$ *satisfies one of the following conditions:*

(a) $\forall x, y \in \mathrm{dom}_{\mathbb{Z}}\, f,\ \exists z \in \mathrm{dom}_{\mathbb{Z}}\, f:\ z \leq x,\ z \leq y,$
(b) $\forall x, y \in \mathrm{dom}_{\mathbb{Z}}\, f,\ \exists z \in \mathrm{dom}_{\mathbb{Z}}\, f:\ z \geq x,\ z \geq y.$

Then, f is M$^\natural$-concave if and only if f satisfies (L1[$\mathbb{Z}$]) *and* (L2[$\mathbb{Z}$]).
(ii) *Suppose that* $\mathrm{dom}_{\mathbb{Z}}\, f$ *is an integral polymatroid. Then, f is M$^\natural$-concave if and only if f satisfies* (L1[$\mathbb{Z}$]) *and* (L2[$\mathbb{Z}$]).

Proof The polymatroid case in claim (ii) follows immediately from the general case (i) since the condition (a) in (i) holds for an integral polymatroid. Hence, it suffices to prove the claim (i). It is shown in the proof of Theorem 1.2 in Section 3 that (L1[$\mathbb{Z}$]) and (L2[$\mathbb{Z}$]) imply (L3[$\mathbb{Z}$]) if $\mathrm{dom}_{\mathbb{Z}}\, f$ satisfies (a) or (b). Thus, the claim (i) follows from Theorem 1.1. □

4.3 Local Exchange Property and Well-Layered Map

The local exchange property (L2[$\mathbb{Z}$]) in (M$^\natural$-EXC$[\mathbb{Z}]_{\mathrm{loc}}$) is deeply related with the concept of well-layered map of Dress–Terhalle [3] (see also [18]).

For a set function $f : 2^N \to \mathbb{R}$ with $\mathrm{dom}\, f = 2^N$, we consider the following incremental greedy algorithm:

Algorithm IncrementalGreedy
Step 0: Set $k := 0$, $X_0 := \emptyset$.
Step 1: Let $i_k \in N$ be an element maximizing the value $f(X_k + i_k)$.
Step 2: Set $X_{k+1} := X_k + i_k$, $k := k + 1$.
If $k > n$, then stop; otherwise, go to Step 1. □

A set function f is called a well-layered map[2] if the sets $X_0, X_1, \ldots, X_n$ generated by the algorithm IncrementalGreedy applied to f satisfies the following condition:

$$X_k \in \arg\max\{f(Y) \mid Y \subseteq N,\ |Y| = k\} \qquad (k = 0, 1, \ldots, n).$$

It is shown [18, Sec. 3] that a set function f is a well-layered map if and only if f satisfies the following local exchange property:

$$\textbf{(L2}[\mathbb{B}]) \ f(X + i + j) + f(X + k) \leq \max\left[f(X + i + k) + f(X + j),\ f(X + j + k) + f(X + i)\right],$$

which is nothing but the property (L2[$\mathbb{Z}$]) specialized to set functions.

[2] In [3] the concept of well-layered map is defined for more general set functions for which the effective domain can be a proper subset of 2^N. We here restrict our attention to set functions f with $\mathrm{dom}\, f = 2^N$ for simplicity of the description.

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