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On the Nonlinear Relationship between Inflation and Growth: A Theoretical Exposition*

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Abstract

This study introduces cash-in-advance constraints into an R&D-based model of endogenous growth in which agents' abilities to develop new goods are heterogeneous. We demonstrate that the negative effect of inflation on long-term growth is weaker in the heterogeneous ability economy than in the homogeneous ability economy if the inflation rate is relatively low, whereas the opposite outcome holds in the high inflation regime. Our numerical examples show that the threshold level of inflation is about 20% per year, which fits well with the findings of existing empirical studies of the nonlinear relation between inflation and growth.

Keywords: endogenous growth, cash-in-advance constraints, heterogeneous agents, nonlinear relationship

JEL Classification: E40, O42

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1 Introduction

Whether inflation harms economic growth is a long-standing question in macroeconomics. Although the issue has not yet been completely settled, empirical studies of inflation and growth have reached broad consensus. The majority of empirical research has found that the correlation between inflation and growth is not prominent when the rate of inflation is relatively low. This finding is the main reason why recent research on monetary macroeconomics mostly ignores the growth effect of inflation and instead focuses on how monetary policy affects short-run business cycles in the presence of price rigidity.[1] At the same time, a number of empirical studies have found a significant negative relation between inflation and growth if the rate of inflation exceeds a threshold level. In other words, the existing empirical findings suggest that a nonlinear relationship between inflation and growth may exist.

Researchers using monetary growth models generally conclude that the theoretical effects of inflation on growth are ambiguous. In fact, one can construct monetary growth models that display various patterns of links between inflation and growth.[2] For example, in the context of a representative agent model with endogenous growth and fixed labor supply, if money is introduced via a cash-in-advance (CIA) constraint on consumption, then the long-term growth rate is insensitive to inflation. While such a model may give rise to a negative growth effect of inflation if a choice between labor and leisure is allowed, its quantitative impact is generally small, as shown by Gomme (1993), Jones and Manuelli (1995), and Chari et al. (1995). On the contrary, it is also possible to construct a model that involves a substantial negative growth effect of inflation if one assumes that inflation tax directly affects investment expenditure on physical capital, human capital, and R&D activities. Therefore, it is easy to derive each regime of the inflation–growth relationship based on a different analytical setting. From a theoretical perspective, this is not a satisfactory treatment of

[1] Another reason is that monetary business cycle models with flexible prices usually show that introducing money does not alter business cycle patterns in a quantitatively significant manner (see Cooley and Hansen, 1989).

[2] The literature on money and growth in the 1960s and 1970s employed neoclassical (exogenous) growth models, focusing on the long-run effect of inflation on the steady-state level of income (see, for example, Sidrauski (1967), Tobin (1965), and Stein (1971). Most studies in the 1990s used endogenous growth models to discuss the long-run impact of inflation on the growth rate of real income.

the growth effect of inflation. The theoretical challenge is thus to describe the nonlinearity of the inflation–growth relationship in a single model. This paper presents such an analytical setting.

The analytical framework of this study is an R&D-based model of endogenous growth in which an expansion in the variety of intermediate goods sustains continuing growth. The final good is produced by using a variety of intermediate goods, each of which is produced by a monopolistically competitive firm. R&D activities expand the variety of intermediate goods by adding new goods. We assume that each agent has different levels of ability of developing new intermediate goods and that there is an endogenously determined cutoff level of ability. In this setting, agents with abilities above the cutoff level become entrepreneurs and inventors, while agents with abilities below the cutoff give up innovation and become workers. Money is introduced via CIA constraints on consumption as well as on expenditure for the fixed and variable costs of producing intermediate goods. The monetary authority is assumed to control the nominal interest rate that directly affects the rate of inflation.

In our model, owing to the presence of the CIA constraint on intermediate good production, a higher inflation rate depresses the monopolistic profits earned by intermediate good firms. This in turn lowers the benefits of R&D represented by the present value of a sum of the monopolistic profits obtained by producing new goods. If agents are homogeneous, such a negative impact of inflation uniformly reduces the incentive for R&D, which lowers economic growth. As a result, the negative relation between inflation and growth is roughly linear in the homogeneous ability economy. If agents' abilities are heterogeneous, the rate of inflation affects the occupational choice condition of agents. When selecting their occupation, agents compare the marginal benefit of being a worker (i.e., the real wage rate) with the marginal benefit of becoming an entrepreneur. As mentioned above, a rise in inflation lowers the marginal benefit of R&D, meaning that the cutoff level of ability for being an entrepreneur rises. If the rate of inflation is relatively low, the rise in the cutoff level makes agents with relatively low abilities give up R&D and become workers. Since such an impact is relatively small, a rise in inflation does not yield a significant negative effect on growth. By contrast, if the rate of inflation is high, an additional rise in inflation generates occupational changes for agents with high abilities. Hence, its negative impact on growth is large.

In this study, we confirm our intuition both analytically and numerically. Under a given distribution function of agents' abilities, we analytically reveal that if the rate of inflation is relatively low, higher inflation yields a negative impact on growth that is weaker in the heterogeneous ability economy than in the homogeneous ability economy. By contrast, if the rate of inflation is relatively high, the growth rate of the heterogeneous ability economy exhibits a stronger negative response than that of the homogeneous ability economy. Assuming that entrepreneurial ability follows a truncated Pareto distribution, we then numerically examine the nonlinear relation between inflation and growth. Our numerical examples show that under plausible parameter values, there is a weak negative relation between inflation and growth if the rate of inflation is less than about 20%. However, if the distribution function of abilities has a "long and fat" tail, there is a sharp decline in the long-run growth rate of income when the rate of inflation exceeds that threshold level.

In our analytical and numerical investigations, we focus on the key elements that give rise to a nonlinear relation between inflation and growth. As for the negative effect of inflation on growth, we show that the CIA constraint on intermediate good production (in particular, the constraint on the expenditure for the fixed cost) plays a relevant role. In addition, we find that the distribution function of ability should have a "long and fat" tail to obtain prominent nonlinearity. We discuss those elements in detail. Hence, our contribution is not only obtaining an empirically plausible inflation–growth linkage but also clarifying the mechanics that generate such a relationship.

Related Literature

(i) Empirical Studies

A number of empirical investigations of the nonlinear relationship between inflation and growth have been conducted. Earlier studies such as Fischer (1993) and Barro (1996, 1997) pointed out the nonlinearity of the correlation between inflation and growth. Fischer (1993) suggests that the overall effects of inflation on growth are negative and that the effects are particularly prominent when the rate of inflation is high. Based on a series of cross-country studies, Barro (1996, 1997) finds that the link between inflation and income growth is not prominent if the inflation rate is below 20% per year. However, if the rate of inflation is relatively high, a 10% increase reduces the growth rate of real GDP by between 0.2% and 0.3%. Bruno and Easterly (1996, 1998) also reveal that

the correlation between inflation and growth is almost negligible in countries with moderate rates of inflation, while the relation is unambiguously negative in high-inflation countries. For them, the critical threshold is an inflation rate of around 40%. Sarel (1996) and Ghosh and Phillips (1998) find structural break points. According to Sarel (1996), the marginal negative effect of inflation on growth is much stronger if the annual rate of inflation is above 8%. Ghosh and Phillips (1998) find that, on average, a rise in inflation from 10% to 20% reduces the growth rate by 0.3–0.4% and a rise in inflation from 20% to 40% lowers the growth rate by 0.8%. Although these empirical studies in the 1990s revealed similar profiles for the inflation–growth relationship, they found various threshold rates of inflation depending on the data sets employed.[3]

Recent studies of the threshold effect of inflation on growth have conducted more sophisticated econometric evaluations. For example, Khan and Senhadji (2001) use the threshold estimation technique and find that threshold inflation levels are 1–3% for developed countries, 7–11% for developing countries, and 8–12% for all countries. Similarly, Kremer et al. (2013) re-examine the relationship between inflation and growth for 40 countries between 1960 and 2004. They find that in the absence of regime intercepts, the threshold level of inflation is 19% and the inclusion of a regime intercept decreases the threshold from 19% to 12%. In addition, Omey and Kan (2010) employ the panel smooth transition regression method that takes account of the nonlinearities in the data. By using a panel data set for six industrialized countries, they find a statistically significant negative and nonlinear relation between inflation and growth. López-Villavicencio and Mignon (2011) also use the same technique and a wider data set to find that the threshold value of the inflation rate strongly differs among advanced and developing countries. Their study shows that the estimated threshold rate of inflation is 2.7% for industrialized economies and 17.5% for emerging ones. Moreover, for inflation rates of around 3%, the inflation–growth link is positive in advanced economies, while it is nonsignificant in developing countries below a 17.5% inflation rate level.[4]

In sum, the empirical studies carried out over the past two decades have clearly demonstrated the negative and nonlinear impact of inflation on economic

[3]See Temple (2000) for a critical evaluation of the empirical studies of inflation and growth conducted in the 1990s.

[4]For an estimation of the threshold rate of inflation, see also Eggoh and Khan (2014), Pollin and Zhu (2006), and Rousseau and Wachtel (2002).

growth. However, these empirical studies do not directly estimate monetary growth models: they simply estimate the reduced form of the inflation–growth relationship by using various data sets. Similarly, our model does not intend to support the specific set of empirical findings obtained so far. Our primary concern is to demonstrate that introducing agent heterogeneity would be helpful to show the presence of the threshold effect of inflation on growth.

(ii) Theoretical Studies

From a theoretical perspective, our study belongs to the literature on money and endogenous growth that was actively studied in the 1990s. Most earlier studies of this topic utilized endogenous growth models with production externalities or models with human capital accumulation. As mentioned earlier, many of these concluded that inflation has a long-run negative effect on growth.[5] More recent studies of money and endogenous growth have focused on R&D-based growth models. Among others, Chu and Cozzi (2014) introduce CIA constraints on consumption and R&D expenditure in a quality ladder model of endogenous growth, while Chu and Lai (2013) analyze a similar model in which money is introduced by using a money-in-the-utility function approach. Huang et al. (2013) also introduce CIA constraints into the quality ladder model of growth, while Oikawa and Ueda (2015) explore the optimal rate of inflation in a similar setting. On the contrary, Chu et al. (2012) and He (2015) introduce money into variety expansion models of endogenous growth.[6] As pointed out by He (2015), the effects of monetary policy on growth and welfare in R&D-based models do not stem from the modelling strategy (i.e., quality ladder vs. variety expansion) but rather from the differences in the way in which money is introduced into the model. Although their model structures are similar to ours, the nonlinear relationship between inflation and growth is out of touch in those studies.[7]

[5]For example, by using two-sector endogenous growth models with physical and human capital, Gomme (1993), Jones and Manuelli (1995), and Mino (1997) show that inflation has a negative impact on the balanced growth rate of the economy. Note that Van der Ploeg and Alogoskoufis (1994) and Mino and Shibata (1995) derive a positive relation between inflation and the balanced growth rate by using overlapping generations models with money.

[6]Among the studies in the 1990s, Marquis and Reffett (1994) explore a variety expansion model of endogenous growth with money.

[7]Another difference between earlier studies and recent ones of monetary endogenous growth models is that most earlier studies assumed that the central bank controls the growth rate of the nominal money stock, whereas the recent literature usually assumes that the central bank controls the nominal interest rate. This study follows the recent approach to monetary policy formulation.

Some authors derive various patterns for the inflation–growth relationship in a single model. Vaona (2012) examines an endogenous growth version of the New Keynesian model and shows that the effect of inflation on growth can be either negligible or sufficiently negative depending on the value of the elasticity of labor supply. Similarly, Chen (2015) constructs a transaction cost-based monetary endogenous growth model with monopolistic competition and reveals various patterns of inflation–growth relations by changing the value of the elasticity of substitution between intermediate inputs. Both authors reveal that the relation between inflation and growth may be hump-shaped if certain conditions are met. While the research interest of those authors overlaps with ours, we obtain an empirically plausible nonlinear relation between inflation and growth *under the given parameter values in the model.*

As for the modeling strategy, our study is closely related to Jaimovich and Rebelo (2016). These authors examine the growth effect of income tax in an R&D-based, nonmonetary endogenous growth model with heterogeneous agents. They numerically derive a nonlinear relation between the rate of income tax and long-run growth rate of the economy. In their numerical experiments, the rate of income tax has little impact on the long-run growth rate of income before it reaches about 60%, whereas it has a significant negative effect on growth if the tax rate exceeds 60%. Their model mimics the weak correlation between taxation and growth found by the foregoing cross-country studies. Jaimovich and Rebelo (2016) also claim that a significant negative relation between taxation and growth in the high-income taxation regime supports our intuition: it is implausible to assume that the long-run growth rate will not decline even if the rate of income tax is 100%. In this sense, Jaimovich and Rebelo's primary concern is to present a model that reconciles the empirical facts with a thought experiment about the growth effects of extremely high rates of income tax. By contrast, our study intends to present a theoretical exposition of the empirically confirmed relationship between inflation and growth.

The remainder of the paper is organized as follows. The next section sets up the model. Section 3 analyzes the relationship between inflation and growth. This section also compares our theoretical results with those obtained in the homogeneous ability economy to derive the conditions for the nonlinear relationship between inflation and growth in the heterogeneous ability economy. Section 4 presents numerical examples of our analytical results and shows that

under plausible parameter values, heterogeneity in entrepreneurial ability produces the empirically plausible nonlinearity between inflation and growth. Section 5 concludes.

2 The Model

Time is continuous and is denoted by $t \geq 0$. A single final good is produced by using labor and intermediate goods. The number of intermediate goods at time t is N_t. Intermediate goods are produced by monopolistically competitive firms. N_t expands through R&D activities, which drives economic growth, as in Romer (1990) and Grossman and Helpman (1993).

We consider a representative "large" household composed of heterogeneous agents. This setting avoids the complexity involved in managing the distribution of money holdings.[8] There is a unit continuum of identical households. The representative "large" household consists of a continuum of agents whose number is constant at L. Following Jaimovich and Rebelo (2016), we assume that agents in the representative household are heterogeneous in their entrepreneurial ability, $h \in [h_{min}, h_{max}]$, where h follows a cumulative distribution $F(h)$ that is continuously differentiable and satisfies $F(h_{min}) = 0$ and $F(h_{max}) = 1$ ($0 < h_{min} < h_{max}$). Agents with the same ability are identical. There is an occupational choice, as in Lucas (1978). Each agent becomes an entrepreneur or a worker. If an agent becomes an entrepreneur, he/she engages in R&D activities to increase the number of intermediate good firms that he/she owns.

2.1 Final Good Production

The production technology of the final good is given by

$$Y_t = l_t^{\alpha} \cdot \int_0^{N_t} z_{j,t}^{1-\alpha} \mathrm{d}j, \tag{1}$$

where Y_t is the final good output, l_t is labor input, $z_{j,t}$ is the quantity of intermediate input $j \in [0, N_t]$, and $\alpha \in (0, 1)$ represents the inverse of the elasticity of substitution among intermediate inputs.

[8] Appendix A presents an alternative setting that generates the same result.

The final good sector is competitive. Profit maximization yields

$$\frac{p_{j,t}}{P_t} = (1-\alpha)\cdot\left(\frac{l_t}{z_{j,t}}\right)^{\alpha}, \quad \forall j, \tag{2}$$

$$w_t = \alpha\int_0^{N_t}\left(\frac{l_t}{z_{j,t}}\right)^{\alpha-1}\mathrm{d}j, \tag{3}$$

where P_t and $p_{j,t}$ are the prices of the final good and intermediate good j, respectively. w_t is the wage rate in terms of the final good.

2.2 Monetary Authority

The monetary authority controls nominal interest rate i_t, which is kept constant over time ($i_t = i > 0\ \forall t$). It rebates seigniorage revenue to households through lump-sum transfers. Then, $T_t = \dot{M}_t/P_t$ holds, where T_t is the lump-sum transfer at time t and M_t is the nominal money stock.

2.3 Households

The utility of the representative "large" household at time s is given by

$$U_s = \int_s^{\infty}\frac{(c_t)^{1-\sigma}-1}{1-\sigma}\cdot e^{-\rho(t-s)}\mathrm{d}t, \tag{4}$$

where c_t denotes the final good consumption per agent at time t. $\rho > 0$ is the time preference rate. $\sigma > 0$ is the inverse of the elasticity of intertemporal substitution. When $\sigma = 1$, instantaneous utility takes a logarithmic form.

Each agent in the representative "large" household owns intermediate good firms. The number of intermediate good firms that a typical agent with ability h owns is $n_{h,t}$, meaning that $N_t = \int_{h_{min}}^{h_{max}} n_{h,t}L\mathrm{d}F(h)$. If an agent with ability h becomes an entrepreneur and engages in R&D for time interval dt, he/she can invent $\delta K_t h\cdot\mathrm{d}t$ new intermediate goods. The presence of K_t represents the knowledge spillover. The law of motion for $n_{h,t}$ is given by $\dot{n}_{h,t} = \delta K_t h\cdot(1-I_{h,t})$, where $I_{h,t} = 1$ holds if an agent with ability h becomes a worker at time t. Otherwise, $I_{h,t} = 0$ holds. The inventor of a new intermediate good can hold a

permanent patent for a newly invented good. N_t evolves according to

$$\dot{N}_t = \int_{h_{min}}^{h_{max}} \dot{n}_{h,t} L \mathrm{d}F(h) = \delta K_t \int_{h_{min}}^{h_{max}} h \cdot (1 - I_{h,t}) L \mathrm{d}F(h). \tag{5}$$

The intermediate good sector is monopolistically competitive. The production of a unit of intermediate good j requires $\eta > 0$ units of the final good as variable costs. In addition, to operate an intermediate good firm, $\xi > 0$ units of the final good are needed as fixed costs. If we use (2), the operating profit of intermediate good j is given by

$$\pi_{j,t} = \left(\frac{p_{j,t}}{P_t} - \eta \right) \cdot z_{j,t} - \xi = (1 - \alpha) \cdot l_t^{\alpha} z_{j,t}^{1-\alpha} - \eta z_{j,t} - \xi. \tag{6}$$

If an agent becomes a worker, he/she earns labor income w_t. Each agent receives profits $\pi_{j,t}$ from the intermediate good firm that he/she owns. The sum of the profit income that an agent with ability h receives is $\pi_{h,t} \equiv \int_{j \in \Xi_{h,t}} \pi_{j,t} \mathrm{d}j$, where $\Xi_{h,t}$ is the set of intermediate good firms that an agent with ability h owns. The representative household as a whole receives $\int_{h_{min}}^{h_{max}} \pi_{h,t} L \mathrm{d}F(h) = \int_0^{N_t} \pi_{j,t} \mathrm{d}j$. The flow budget constraint of the representative "large" household is

$$c_t L + \dot{b}_t + \dot{m}_t = r_t b_t + \int_{h_{min}}^{h_{max}} w_t I_{h,t} L \mathrm{d}F(h) + \int_0^{N_t} \pi_{j,t} \mathrm{d}j + T_t - \mu_t m_t, \tag{7}$$

where b_t and $m_t (\equiv M_t / P_t)$ denote the real bond and real money holdings of the representative household, respectively. The variables r_t and $\mu_t \equiv \dot{P}_t / P_t$ denote the real interest rate and inflation rate, respectively.

A fraction $\theta_c \in [0, 1]$ of consumption expenditure is subject to a CIA constraint. In addition, a fraction $\theta_\eta \in [0, 1]$ of the variable cost and a fraction $\theta_\xi \in [0, 1]$ of the fixed cost must be financed by money.[9] The CIA constraint is given by

$$m_t \geq \theta_c c_t L + \theta_\eta \int_0^{N_t} \eta z_{j,t} \mathrm{d}j + \theta_\xi N_t \xi. \tag{8}$$

Given b_0, m_0, $n_{h,0}$, and N_0, the representative household maximizes (4) sub-

[9]Here, we assume that a part of the final good should be purchased by paying cash. We may assume that intermediate goods are also cash goods; however, as we see below, such an assumption does not play an essential role in deriving the nonlinear relation between inflation and growth.

ject to (5), (6), (7), and (8). The first-order conditions are given by

$$
\begin{aligned}
c_t &: (c_t)^{-\sigma} = (\lambda_t + \theta_c \psi_t) L, && \text{(9a)}\\
z_{j,t} &: \lambda_t \left\{ (1-\alpha)^2 l_t^{\alpha} z_{j,t}^{-\alpha} - \eta \right\} = \psi_t \theta_\eta \eta, && \text{(9b)}\\
b_t &: \dot{\lambda}_t = (\rho - r_t)\lambda_t, && \text{(9c)}\\
m_t &: -\lambda_t \mu_t + \psi_t = -\dot{\lambda}_t + \rho \lambda_t, && \text{(9d)}\\
N_t &: \lambda_t \cdot \left\{ (1-\alpha) l_t^{\alpha} z_{N_t,t}^{1-\alpha} - \eta z_{N_t,t} - \xi \right\} - \psi_t \cdot (\theta_\eta \eta z_{N_t,t} + \theta_\xi \xi) = -\dot{\zeta}_t + \rho \zeta_t, && \text{(9e)}\\
I_{h,t} &: I_{h,t} = \begin{cases} 1 & \text{if } \lambda_t w_t > \zeta_t \delta K_t h, \\ 0 & \text{if } \lambda_t w_t \leq \zeta_t \delta K_t h, \end{cases} && \text{(9f)}
\end{aligned}
$$

where λ_t, ζ_t, and ψ_t are the costate variables associated with the budget constraint, law of motion for N_t, and CIA constraint, respectively. The following discussion assumes $K_t = N_t$, as is common in the literature (see Grossman and Helpman, 1993).

CIA Constraint and Euler Equation

From (9c), (9d), and the Fisher equation ($i = r_t + \mu_t$), we obtain

$$\psi_t = i\lambda_t > 0. \tag{10}$$

Equations (9a) and (10), together with $c_{h,t} > 0$, imply $\psi_t > 0$ for all $t \geq 0$. As a result, the CIA constraint is always binding.

From (9a), (9c), and (10), we obtain the following consumption Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} \cdot (r_t - \rho). \tag{11}$$

Intermediate Good Production

From (9b) and (10), we know that intermediate good firm j produces

$$z_{j,t} = \left[\frac{(1-\alpha)^2}{\eta \cdot (1 + i\theta_\eta)} \right]^{\frac{1}{\alpha}} \cdot l_t \equiv z_t. \tag{12}$$

Since all producers choose the same quantity, we eliminate the subscript j from $z_{j,t}$ in what follows. By using (10) and (12), we rewrite (9e) as

$$\lambda_t \cdot \underbrace{\left\{(1-\alpha)l_t^{\alpha} z_t^{1-\alpha} - (1+i\theta_\eta)\eta z_t - (1+i\theta_\xi)\xi\right\}}_{\widehat{\pi}_t} = -\dot{\zeta}_t + \rho\zeta_t,$$

where $\widehat{\pi}_t$ represents the profits of an intermediate good firm net of the costs generated by the CIA constraint: $\widehat{\pi}_t = \pi_t - i\theta_\eta\eta z_t - i\theta_\xi\xi$. Let us define $\nu_t = \zeta_t/\lambda_t$. The above equation, together with (9c), implies $r_t\nu_t = \dot{\nu}_t + \widehat{\pi}_t$, which has the following solution:

$$\nu_t = \int_t^{\infty} \widehat{\pi}_\tau e^{-\int_t^\tau r_u du} d\tau. \tag{13}$$

Thus, ν_t can be interpreted as the value of an intermediate good firm.

Occupational Choice

Equation (9f) implies that threshold ability h_t^* makes agents indifferent between being a worker and being an entrepreneur. From (9f), h_t^* satisfies

$$w_t = \nu_t \cdot \delta N_t h_t^*. \tag{14}$$

The left-hand side (LHS) shows the opportunity cost of being an entrepreneur, while the right-hand side (RHS) shows the benefit of being an entrepreneur. An increase in i affects $\widehat{\pi}_t$ and ν_t and hence influences the benefit of being an entrepreneur.

From (9f) and (14), agents with an ability above h_t^* become entrepreneurs and the others become workers. The number of entrepreneurs at time t is $L \cdot \{1 - F(h_t^*)\}$ and the number of workers (the labor supply for final good production) at time t is $l_t = LF(h_t^*)$. Hence, (5) can be written as $\dot{N}_t = \delta N_t LH(h_t^*)$, where $H(h_t^*) \equiv \int_{h_t^*}^{h_{max}} h\mathrm{d}F(h)$. The growth rate of N_t is given by

$$\frac{\dot{N}_t}{N_t} = \delta LH(h_t^*) \equiv g(h_t^*). \tag{15}$$

2.4 Equilibrium Conditions

The following four equilibrium conditions for the economy exist:

$$b_t = 0, \tag{16a}$$

$$\frac{M_t}{P_t} = \theta_c L c_t + N_t(\theta_\eta z_t + \theta_\xi \xi), \tag{16b}$$

$$Y_t = L c_t + N_t \eta z_t + N_t \xi, \tag{16c}$$

$$L = F(h_t^*)L + (1 - F(h_t^*))L = l_t + (1 - F(h_t^*))L. \tag{16d}$$

The equilibrium condition for the credit market is given by (16a). Since we assume a closed economy, the net supply of real bonds is zero. The equilibrium conditions for the money market, final good market, and labor market are given by (16b), (16c), and (16d), respectively.

2.5 The Dynamics of h_t^*

Appendix B shows that the dynamics of threshold ability h_t^* are given by

$$\frac{\dot{h}_t^*}{h_t^*} = \frac{\sigma}{1 + \sigma\Omega(h_t^*; i)} \cdot \left[\frac{1}{\sigma} \cdot \left\{ \frac{h_t^* \{\Pi F(h_t^*) - \Phi(i)\}}{\Gamma} - \rho \right\} - g(h_t^*) \right], \tag{17}$$

where

$$\begin{aligned}
\Omega(h_t^*; i) &\equiv \frac{1}{\widehat{c}_t} \cdot \left[\frac{(1-\alpha)^2}{\eta(1 + i\theta_\eta)} \right]^{\frac{1-\alpha}{\alpha}} \cdot \left[1 - \frac{(1-\alpha)^2}{1 + i\theta_\eta} \right] \cdot F'(h_t^*) h_t^*, \\
g(h_t^*) &\equiv \delta L H(h_t^*), && \text{(18a)} \\
\Phi(i) &\equiv \left\{ (1 + i\theta_\eta)\eta \right\}^{\frac{1-\alpha}{\alpha}} (1 + i\theta_\xi)\xi, && \text{(18b)} \\
\Pi &\equiv \alpha(1-\alpha)^{\frac{2-\alpha}{\alpha}} L, && \text{(18c)} \\
\Gamma &\equiv \frac{1}{\delta} \cdot \alpha(1-\alpha)^{\frac{2(1-\alpha)}{\alpha}}.
\end{aligned}$$

$\Phi(i)$ represents the sum of the production costs and CIA costs.

2.6 The Steady-State Equilibrium

To study the steady-state equilibrium, we assume the following two conditions.

Assumption 1. $h_{max} \{\Pi \cdot F(h_{max}) - \Phi(0)\} > \rho\Gamma$.

Assumption 2. At least one of $\theta_\eta \in [0,1]$ and $\theta_\xi \in [0,1]$ is strictly positive.

Assumption 1 holds if L is sufficiently large (see (18c)). This assumption ensures the existence of a steady state with positive growth. Assumption 2 means that at least one of the variable cost and the fixed cost of intermediate good production is subject to the CIA constraint. Assumption 2 ensures that $\Phi(i)$ is an increasing function of i and $\lim_{i\to\infty}\Phi(i) = \infty$. Thus, there exists a unique $i_{max} > 0$ that satisfies

$$h_{max}\{\Pi \cdot F(h_{max}) - \Phi(i_{max})\} = \rho\Gamma. \tag{19}$$

We define a steady-state equilibrium as an equilibrium where h_t^* is constant over time. Hereafter, the variables without subscript t denote steady-state values. Under Assumptions 1 and 2, we can prove the next proposition.

Proposition 1 *Suppose that Assumptions 1 and 2 hold. For $i \in (0, i_{max})$, the economy always stays in a unique steady-state equilibrium where the growth rate of N_t, $g(i)$, is strictly positive and Y_t and c_t grow at the same rate as N_t.*

Proof. See Appendix C.

Since the growth rate, inflation rate, and threshold ability are functions of the nominal interest rate in the steady state, we denote them by $g(i)$, $\mu(i)$, and $h^*(i)$, respectively. The continuous differentiability of $F(h)$ ensures that $g(i)$, $\mu(i)$, and $h^*(i)$ are also continuously differentiable. Note that i_{max} is the upper bound nominal interest rate that ensures positive growth, $g(i) > 0$, and that $g(i_{max}) = 0$ (see Appendix C). From (11) and $i = r_t + \mu_t$, $\mu(i)$ is given by

$$\mu(i) = i - r = i - \rho - \sigma g(i). \tag{20}$$

The growth rate of nominal money $\dot{M}_t/M_t$ is equal to $\mu(i) + g(i)$ (see Appendix C).

Before studying the relationship between inflation and growth, we examine the effects of i on $g(i)$ and $\mu(i)$. By using (18a), Appendix D shows

$$\frac{\mathrm{d}g(i)}{\mathrm{d}i} = -\frac{\delta L h^*(i)\Phi'(i)}{\Pi + \sigma\Gamma\delta L + \dfrac{\Gamma(\rho+\sigma g)}{(h^*(i))^2 F'(h^*(i))}} \equiv \Delta_{i,g}(i) < 0. \tag{21}$$

If Assumption 2 is not satisfied, we have $\Phi'(i) = 0$. Then, the nominal interest rate has no growth effect. Assumption 2 ensures $\Phi'(i) \neq 0$. Thus, the nominal interest rate has a growth effect. Under Assumption 2, an increase in i decreases $g(i)$. The intuition is simple. Since an increase in i tightens the CIA constraint, the net profit, $\widehat{\pi}_t$, and value of an intermediate good firm, v_t, decrease, which lowers the benefits of being an entrepreneur (see (B.5) in Appendix B and (13)). Then, the threshold ability increases, $\mathrm{d}h^*(i)/\mathrm{d}i > 0$ (see Appendix C). The number of entrepreneurs is negatively affected and thus the growth rate is depressed. From (20), we have $\mathrm{d}\mu(i)/\mathrm{d}i = 1 - \sigma\Delta_{i,g}(i) > 0$ because of $\Delta_{i,g}(i) < 0$. Then, an increase in i has a positive effect on the inflation rate.

Note that a strictly positive fixed cost of intermediate good production $\xi > 0$ is necessary for the nominal interest rate to have a growth effect. Let us reconsider (14) that determines the threshold ability, $h^*(i)$. Consider the steady-state equilibrium. Since the real interest rate is constant at the steady state, (13) is rewritten as $v = \frac{\widehat{\pi}}{r}$. By substituting (B.5), (B.6), and $v = \frac{\widehat{\pi}}{r}$ into (14), we obtain

$$\alpha(1-\alpha)^{\frac{2(1-\alpha)}{\alpha}} \left\{(1+i\theta_\eta)\eta\right\}^{-\frac{1-\alpha}{\alpha}} \cdot N_t = \frac{\Pi\left\{(1+i\theta_\eta)\eta\right\}^{-\frac{1-\alpha}{\alpha}} F(h_t^*) - (1+i\theta_\xi)\xi}{r} \cdot \delta N_t h_t^*.$$

The LHS is the opportunity cost of being an entrepreneur, w, while the RHS is the benefit of being an entrepreneur, $v\delta N h^* (= \widehat{\pi}\delta N h^*/r)$. When $\xi = 0$, the above equation is independent of i. Hence, the nominal interest rate has no effect on the threshold ability and has no growth effect. Only when $\xi > 0$ does i affect the threshold ability. The reason is as follows. The benefit of being an entrepreneur is affected by the fixed cost through the net profit, $\widehat{\pi}$. However, the opportunity cost of being an entrepreneur, w, is independent of the fixed cost because w_t equals the *marginal* product of labor. Therefore, in the presence of the fixed cost, i has different effects on both sides of the above equation and thus affects the threshold ability.

3 The Relationship between Inflation and Growth

The discussion in the previous section implies a negative relationship between inflation and growth. Since we are interested in the nonlinearity of this relationship, we examine the magnitude of this relationship. To this end, we totally

differentiate (20) to obtain

$$\mathrm{d}g(i) = \frac{\Delta_{i,g}(i)}{1-\sigma\cdot\Delta_{i,g}(i)}\cdot\mathrm{d}\mu(i) = \Delta_{\mu,g}(i)\cdot\mathrm{d}\mu(i), \tag{22}$$

where $\Delta_{\mu,g}(i) \equiv \frac{\Delta_{i,g}(i)}{1-\sigma\cdot\Delta_{i,g}(i)}$. Since $\Delta_{i,g}(i) \equiv \mathrm{d}g(i)/\mathrm{d}i < 0$, we have $\Delta_{\mu,g}(i) < 0$. Then, there is a negative relationship between inflation and growth. Moreover, a large $|\Delta_{i,g}(i)|$ implies a large $|\Delta_{\mu,g}(i)|$.

3.1 Homogeneous Ability Economy

To highlight the role of heterogeneity in ability, we also consider a homogeneous ability economy in which all agents have the same ability, $\widehat{h} > 0$. Denote the fraction of workers and equilibrium growth rate in the homogeneous ability economy by $q_t \in [0,1]$ and g_t^H, respectively. Hereafter, the variables with superscript H denote those variables for he homogeneous ability economy. We have $g_t^H = \delta\widehat{h}L(1-q_t)$. Appendix E shows that in the steady-state equilibrium, where q_t is constant at $q(i) \in (0,1)$ (see (E.3)), the growth rate is given by

$$g^H(i) \;=\; \delta L\cdot\frac{\widehat{h}(\Pi-\Phi(i))-\Gamma\rho}{\Pi+\sigma\Gamma\delta L}. \tag{23}$$

To ensure $g^H(i) > 0$, i must be below i_{max}^H, where i_{max}^H is defined by $\widehat{h}(\Pi - \Phi(i_{max}^H)) = \Gamma\rho$ (or $g^H(i_{max}^H) = 0$). From (23), we obtain

$$\frac{\mathrm{d}g^H(i)}{\mathrm{d}i} = -\frac{\delta L\widehat{h}\Phi'(i)}{\Pi+\sigma\Gamma\delta L} \equiv \Delta_{i,g}^H(i) < 0. \tag{24}$$

Assumption 2 ensures $\Phi'(i) < 0$. If we replace $g(i)$, $\mu(i)$, $\Delta_{i,g}(i)$, and $\Delta_{\mu,g}(i)$ with $g^H(i)$, $\mu^H(i)$, $\Delta_{i,g}^H(i)$, and $\Delta_{\mu,g}^H(i)$, respectively, (20) and (22) still hold in a homogeneous ability economy. In this homogeneous ability economy, the inflation rate also increases with i and there is a negative relationship between inflation and growth.

As in the heterogeneous ability economy, a strictly positive fixed cost of intermediate good production, $\xi > 0$, is necessary to obtain the growth effect of the nominal interest rate.

3.2 Comparing the Heterogeneous and Homogeneous Ability Economies

We now compare the heterogeneous ability economy with the homogeneous ability economy to highlight the role of heterogeneity. The comparison between (21) and (24) shows two differences between these economies: (i) the third term in the denominator on the RHS of (21) and (ii) the terms $h^*(i)$ and $\widehat{h}$ in the numerator of both equations. The first difference suggests that in the heterogeneous ability economy, if the density of agents with threshold ability $F'(h^*(i))$ is high, $|\Delta_{i,g}(i)|$ becomes large. The nominal interest rate affects growth through its effects on the occupational choices of agents with the threshold ability. Thus, as the number of agents with the threshold ability increases, the nominal interest rate tends to have a large growth effect. The second difference shows that in the heterogeneous ability economy, $|\Delta_{i,g}(i)|$ tends to increase with threshold ability $h^*(i)$. This is because the occupational choices of high-ability agents have larger impacts on growth than those of low-ability agents.

These differences between the homogeneous and heterogeneous ability economies produce different inflation–growth relationships through (22). Then, we obtain the following proposition.

Proposition 2 *Suppose that Assumptions 1 and 2 hold and that there exists* $\bar{i} \in (0, \min\{i_{max}, i^H_{max}\})$ *such that* $g(\bar{i}) = g^H(\bar{i}) > 0$ *holds. Then,* $\bar{i}$ *is unique and for* $i \in (0, \bar{i}]$, *we have* $h^*(i) < \widehat{h}$ *and*

$$\text{(i)} \quad 0 > \Delta_{i,g}(i) > \Delta^H_{i,g}(i), \tag{25}$$

$$\text{(ii)} \quad 0 > \Delta_{\mu,g}(i) > \Delta^H_{\mu,g}(i). \tag{26}$$

Proof. See Appendix F.

Since the inflation rate increases with i, we have $\max\{\mu(i), \mu^H(i)\} \leq \mu(\bar{i}) = \bar{i} - \rho - \sigma g(\bar{i})$ for $i \in (0, \bar{i}]$. Therefore, Proposition 2 suggests that for low inflation, the heterogeneous ability economy has a weaker negative relationship between inflation and growth than the homogeneous ability economy. The intuition is as follows: when the nominal interest rate is low and hence the inflation rate is also low, the CIA constraint is loose. The net profit and value of an intermediate good firm, $\widehat{\pi}_t$ and v_t, are both large. Being an entrepreneur generates large

benefits. Then, in the heterogeneous ability economy, even agents with low ability become entrepreneurs ($h^*(i) < \widehat{h}$ for $i \in (0, \bar{i}]$). When the nominal interest rate and inflation rate increase, these low-ability entrepreneurs switch to being workers, which has a negative effect on growth. However, because the abilities of these agents are low, their occupational choices have only a small impact on growth (see (25)). Then, for low inflation, a weak negative relationship between inflation and growth arises in the heterogeneous ability economy (see (26)).

Furthermore, for a high inflation rate, we prove the next proposition.

Proposition 3 *Suppose that h_{max} is sufficiently large and $\lim_{h_{max}\to+\infty}(h_{max})^3F'(h_{max}) = +\infty$ and that Assumptions 1 and 2 hold. For sufficiently high $i \in (0, i_{max})$, we have*

$$\text{(i)} \quad 0 > \min_{i\in(0,i^H_{max})} \left\{\Delta^H_{i,g}(i)\right\} > \Delta_{i,g}(i), \tag{27}$$

$$\text{(ii)} \quad 0 > \min_{i\in(0,i^H_{max})} \left\{\Delta^H_{\mu,g}(i)\right\} > \Delta_{\mu,g}(i). \tag{28}$$

Proof. See Appendix G.

A high i implies a high inflation rate, $\mu(i)$. Therefore, Proposition 3 implies that for a sufficiently high inflation rate, the heterogeneous ability economy has a stronger negative relationship between inflation and growth than the homogeneous ability economy.

The condition that h_{max} is sufficiently large means a high upper bound of ability. This implies a "long-tailed" distribution of ability. The condition $\lim_{h_{max}\to+\infty}(h_{max})^3F'(h_{max}) = +\infty$ means that there is a nonnegligible number of high-ability agents. This implies a "fat-tailed" distribution of ability. Thus, these two conditions imply a "long and fat-tailed" distribution of ability.

The intuition of Proposition 3 is simple. When the inflation rate is high, only high-ability agents become entrepreneurs. Because of the "long and fat-tailed" distribution of ability, the occupational choices of these high-ability agents have large impacts on growth (see (27)), which results in a strong negative relationship between inflation and growth (see (28)).

As an example of fat-tailed distributions, consider a truncated Pareto distribution with a shape parameter $a(\geq 1)$, a lower bound $h_{min}(> 0)$, and an upper

bound $h_{max}(> h_{min})$:

$$F(h) = \frac{1 - (h_{min}/h)^a}{1 - (h_{min}/h_{max})^a}. \tag{29}$$

With a small shape parameter, $a \in [1, 2)$, this distribution satisfies $\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max}) = +\infty$. If $a \geq 2$, (29) does not satisfy $\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max}) = +\infty$. Uniform distributions also satisfy $\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max}) = +\infty$.

3.3 Nonlinear Relationship between Inflation and Growth

We now establish the nonlinearity between inflation and growth in the heterogeneous ability economy. Suppose that Propositions 2 and 3 hold. Then, (26) and (28) imply the following relation:

$$\begin{aligned} 0 > \Delta_{\mu,g}(i)\Big|_{i\in(0,\bar{i}]} &> \Delta^H_{\mu,g}(i)\Big|_{i\in(0,\bar{i}]} \\ &\geq \min_{i\in(0,i^H_{max})} \left\{\Delta^H_{\mu,g}(i)\right\} \\ &> \Delta_{\mu,g}(i)\Big|_{\text{sufficiently high } i\in(0,i_{max})}. \end{aligned}$$

Note that the inflation rate $\mu(i)$ increases with i. The above relation implies that in the heterogeneous ability economy, the magnitude of the negative relationship between inflation and growth is small for a low inflation rate, while it is large for a high inflation rate. Then, the heterogeneous ability economy has a nonlinear relationship between inflation and growth.

To prove Propositions 2 and 3, we need the following three conditions: (i) there exists $\bar{i} \in (0, \min\{i_{max}, i^H_{max}\})$ such that $g^H(\bar{i}) = g(\bar{i}) > 0$ holds, (ii) h_{max} is sufficiently large, and (iii) $\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max}) = +\infty$. Proposition 2 shows that condition (i) implies $h^*(i) < \widehat{h}$ for $i \in (0, \bar{i}]$. Since $h^*(i) \geq h_{min}$ must hold, condition (i) can be satisfied only if (i′) $h_{min} > 0$ is sufficiently small. Thus, all three conditions are concerned with the distribution of entrepreneurial ability. Heterogeneity in ability plays an important role for generating the nonlinearity between inflation and growth. Conditions (i′) and (ii) suggest a sufficiently large difference between h_{min} and h_{max}. Condition (iii) implies a nonnegligible number of high-ability agents. Consequently, we can conclude that if there is substantial heterogeneity in ability ("long-tailed distribution" of ability) and the number of high-ability agents is nonnegligible ("fat-tailed distribution" of abil-

ity), heterogeneity in ability generates a nonlinear relationship between inflation and growth.

In the homogeneous ability economy, a counterfactual nonlinear relationship may exist. From (24), we have $\text{sign}\{\text{d}\Delta^H_{i,g}(i)/\text{d}i\} = \text{sign}\{-\Phi''(i)\}$, where

$$\text{sign}\{-\Phi''(i)\} = \text{sign}\left\{-\theta_\eta\left[2\theta_\xi + \frac{1-2\alpha}{\alpha}\frac{1+i\theta_\xi}{1+i\theta_\eta}\theta_\eta\right]\right\}.$$

This equation shows that $\text{d}\Delta^H_{i,g}(i)/\text{d}i$ can be positive or negative depending on the parameters. If $\text{d}\Delta^H_{i,g}(i)/\text{d}i$ is positive, $|\Delta^H_{i,g}(i)|$ decreases with i because of $\Delta^H_{i,g}(i) < 0$. Since a small $|\Delta^H_{i,g}(i)|$ implies a small $|\Delta^H_{\mu,g}(i)|$, the magnitude of the negative relationship between inflation and growth becomes small (large) for a high (low) inflation rate, which is inconsistent with the empirical findings. We emphasize that even when the homogeneous case generates a counterfactual nonlinear relationship, the heterogeneous case produces an inflation–growth nonlinearity that is consistent with the empirical findings.

4 Numerical Examples

Section 3 analytically showed that "long and fat-tail-distributed" entrepreneurial ability generates a nonlinear relationship between inflation and growth. This section presents numerical examples to examine whether the nonlinearity between inflation and growth is obtained under plausible parameter values.

4.1 Calibration

We begin with parameter values other than entrepreneurial ability and its distribution. Section 3 showed that the distribution of entrepreneurial ability is important for the nonlinearity between inflation and growth. Therefore, our results concerning nonlinearity are unaffected qualitatively by the choices of parameters other than entrepreneurial ability and its distribution.

We set the strength of the CIA constraint to one ($\theta_c = \theta_\eta = \theta_\xi = 1$). The discussion later uses different values for the strength of the CIA constraint. We assume $\alpha = 0.6$ to ensure that the labor share in the final good sector is 60%. The inverse of the intertemporal elasticity of substitution is set to $\sigma = 2$. We set $\rho = 0.01$ to ensure that the annual real interest rate in an economy with no

growth is 1%. We normalize the constant marginal cost of intermediate good production to one ($\eta = 1$).

Because of $F(h_{max}) = 1$, (19) implies that when $h_{max} \to +\infty$, i_{max} converges to $\overline{i_{max}}$, where $\overline{i_{max}}$ satisfies $\Pi = \Phi(\overline{i_{max}})$, or $\left\{(1 + \overline{i_{max}}\theta_\eta)\eta\right\}^{\frac{1-\alpha}{\alpha}} (1 + \overline{i_{max}}\theta_\xi)\xi = \alpha(1-\alpha)^{\frac{2-\alpha}{\alpha}} L$ (see (18b) and (18c)). Then, when h_{max} is substantially large, the upper bound of the nominal interest rate, i_{max}, that ensures positive growth depends on the value of L/ξ, given $\alpha = 0.6$, $\eta = 1$, and $\theta_c = \theta_\eta = \theta_\xi = 1$. Here, we assume $L/\xi = 20$, which implies that i_{max} is around 23% when h_{max} is sufficiently large. We set the population to one ($L = 1$), which implies $\xi = 0.05$. Even if we set $L = 20$ and $\xi = 1$, the results are unaffected.

We assume that entrepreneurial ability h follows a truncated Pareto distribution, (29), which is a typical example of long and fat-tailed distributions. If $a \in [1, 2)$ ($a \geq 2$), $\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max}) = +\infty$ is (not) satisfied by (29). The distribution of ability is governed by three parameters: a, h_{min}, and h_{max}. As in Jaimovich and Rebelo (2015), we set the lower bound of ability to one ($h_{min} = 1$). We choose the values of a and h_{max} as well as the value of the strength of knowledge spillover δ as follows.

We first guess the value of a. The next step determines the value of a. Given this guess, we set δ and h_{max} to satisfy two empirical facts. The first fact is that the growth rate is 2% when the inflation rate is also 2%, which is roughly consistent with the U.S. observation. Concerning this fact, we use (20) to find the value of i that ensures $g = 0.02$ when $\mu = 0.02$. We denote this value of i as i_{target}. The following procedure sets $i = i_{target}$.

The second fact concerns firm size distribution. According to data taken from the U.S. Census Bureau (http://www.census.gov/econ/susb/), in the U.S. economy, there were 5,726,160 firms and 115,938,468 employments in 2012. The data show that 32,334,931 employments were employed by the largest 964 firms with more than 10,000 employees. This fact implies that the top 0.017% of U.S. firms employ 27.9% of the labor force.

Concerning the second fact, we follow Jaimovich and Rebelo (2013) and assume that the intermediate good sector and final good sector are vertically integrated, meaning that the owners of intermediate good firms (i.e., entrepreneurs) hire workers to produce final goods. We also assume that the initial ownership of intermediate good firms is distributed among entrepreneurs in proportion to

their ability:

$$s_{h,0} \equiv \frac{n_{h,0}}{N_0} = \frac{h}{L\int_{h^*}^{h_{max}} h\mathrm{d}F(h)},$$

where $s_{h,0}$ is the initial share of the intermediate good firms owned by an agent with ability h. Entrepreneurs take this initial distribution as given. Under this assumption, $n_{h,t}$ grows at the same rate as N_t and hence the distribution of ownership becomes time-invariant.[10] Recall that all intermediate good firms produce the same quantity. The number of intermediate goods that an agent owns is proportional to the number of the workers that the agent employs. Therefore, firm size is proportional to the ability of the entrepreneur.

To find the values of h_{max} and δ given the guess of $a(\geq 1)$, we use an iterative process. First, we guess the values of h_{max} and δ and then compute h^* by setting $\dot{h}_t^* = 0$ in (17). Next, we compute $\bar{h}$ that satisfies the following equation:

$$\frac{\int_{\bar{h}}^{h_{max}} \mathrm{d}F(h)}{\int_{h^*}^{h_{max}} \mathrm{d}F(h)} = 0.00017.$$

The value of $\bar{h}$ determines the top 0.017% of firms. The requirement that the top 0.017% of entrepreneurs account for 27.9% of employment is written as

$$\frac{\int_{\bar{h}}^{h_{max}} h\mathrm{d}F(h)}{\int_{h^*}^{h_{max}} h\mathrm{d}F(h)} = 0.279.$$

Given h^* and $\bar{h}$, we compute the value of h_{max} by using the above equation. Then, we compute the value of δ, using (15), to ensure that $g = 0.02$ holds when $i = i_{target}$. We iterate this process until the values of h_{max} and δ converge. If $a = 1$, convergence occurs when $\delta = 0.00174$ and $h_{max} = 58,533,854$. If $a = 1.1$ ($a = 1.5$), we obtain $\delta = 0.00397$ and $h_{max} = 212,592,778$ ($\delta = 0.03425$ and $h_{max} = 718$). If $a \geq 2$, this iterative process does not converge.

Finally, we determine the value of $a(\geq 1)$. According to the data taken from the U.S. Census Bureau (http://www.census.gov/econ/susb/), in the U.S. economy, the number of large firms with more than 500 employees was 18,219 in 2012. This accounts for about 0.3% of all firms. These firms employ 59,875,575

[10] Note that $\dot{n}_{h,t}/n_{h,t} = \delta N_t h/n_{h,t}$. If we substitute $n_{h,t}/N_t = s_h$, we have $\dot{n}_{h,t}/n_{h,t} = g$.

workers, which accounts for 51.6% of all employments. If $a = 1$, our model indicates that the top 0.3% of firms employ 51.6% of workers, which fits well with the U.S. data. If we increase the value of a, the employment share of the top 0.3% of firms decreases. For example, if we set $a = 1.1$ ($a = 1.5$), the top 0.3% of firms account for 45.8% (3.8%) of employment. Thus, we set $a = 1$, which ensures that the distribution of the ability has a fat tail, $\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max}) = +\infty$. Setting $a = 1$ implies $h_{max} = 58,533,854$, which seems to be sufficiently large. Thus, our benchmark calibration based on the U.S. data seems to produce a long and fat-tailed distribution of ability. Finally, under these parameters, we have $h^* = 363.7$ when the inflation rate is 2%. We have $h_{max}/h^* = 160,940$, which means that the size of the largest firm is 160,940 times larger than that of the smallest firm.

Except for $\widehat{h}$, our numerical example for the homogeneous ability economy uses the same parameter values as those used for the heterogeneous ability economy. Given $\alpha = 0.6$, $\sigma = 2$, $\rho = 0.01$, $\eta = 1$, $\xi = 0.05$, $L = 1$, $\theta_c = \theta_\eta = \theta_\xi = 1$, and $\delta = 0.00174$, we set $\widehat{h}$ to ensure that $g^H(i_{target})$ is 2%, which implies that μ^H is also 2%. This yields $\widehat{h} = 416$.

4.2 Results

Figure 1 shows our numerical results. Panels (a) and (b) display the effects of the nominal interest rate on inflation and growth, respectively. By combining these two panels, we obtain the relationships between inflation and growth (Panel (c)). As we showed in Section 3 analytically, in the presence of heterogeneity in ability, the inflation–growth relationship is highly nonlinear, while it is roughly linear without heterogeneity in ability. In the homogeneous ability economy, as inflation increases from 2% to 10%, the growth rate decreases from 2% to 1.1%. For the same changes in inflation, the heterogeneous ability economy experiences a limited reduction in the growth rate from 2% to 1.9%. However, for inflation above 18%, the heterogeneous ability economy exhibits a much stronger negative relationship. As inflation increases from 18% to 21%, the growth rate decreases sharply from 1.6% to 0.5%.

Panel (d) shows the negative relationships between inflation and the fraction of entrepreneurs. In both the homogeneous and the heterogeneous ability economies, these relationships are roughly linear. However, the heterogeneous

ability economy exhibits a highly nonlinear relationship between inflation and growth. This is because as inflation increases, the ability of entrepreneurs who exit increases. Panel (d) also shows the following point. In the homogeneous ability economy, labor reallocation from the production sector to the R&D sector is an important mechanism behind the inflation–growth relationship. However, such a labor reallocation is limited in the heterogeneous ability economy. Instead, increases in threshold ability h^* derive the nonlinearity of the inflation–growth relationship.

[Figure 1]

4.3 Welfare Implications

The nonlinearity between inflation and growth provides important welfare implications. Since the economy is always in the steady-state equilibrium, (4) can be written as

$$U_0 = \frac{(c_0)^{1-\sigma}}{(1-\sigma)[\rho + (\sigma-1)\widetilde{g}]} = \frac{(\widehat{c_0}N_0)^{1-\sigma}}{(1-\sigma)[\rho + (\sigma-1)\widetilde{g}]},$$

where $\widetilde{g} = g(i)$ or $g^H(i)$. In the heterogeneous ability economy, $\widehat{c_t}$ is defined by (B.2). For the homogeneous ability economy, $F(h^*)$ in (B.2) must be replaced by the fraction of workers q, which is given by (E.3). We normalize N_0 to one and then we have $c_0 = \widehat{c_0}$.

Figure 2 plots the welfare costs against inflation. Panels (a) and (b) Show the percentage changes in U_0 and welfare losses in terms of consumption, respectively, when the inflation rate changes from 2%.[11]

[Figure 2]

Both panels demonstrate that in the heterogeneous ability economy, low inflation is associated with relatively low welfare costs. When inflation rises from 2% to 3%, the homogeneous ability economy experiences a 4.1% decrease in welfare. By contrast, in the presence of heterogeneity, welfare decreases by only

[11] We calculate these welfare losses in terms of consumption as follows. Denote the consumption level when inflation is 2% by $c_{0,2}$. In addition, we denote the welfare level when inflation is x% by $U_{0,x}$. By assuming that the inflation rate is 2%, or equivalently, the growth rate is 2%, we calculate the initial consumption level $c_{0,x}$ that achieves the same welfare as $U_{0,x}$. After that, the percentage difference between $c_{0,2}$ and $c_{0,x}$ is computed.

0.87%. In terms of consumption, the welfare loss is 3.94% in the homogeneous ability economy, while it is 0.86% in the heterogeneous ability economy. At the same time, the two panels show that in the heterogeneous ability economy, high inflation is associated with significantly large welfare costs. In the presence of heterogeneous ability, when inflation increases from 20% to 22%, the welfare loss rises from 78% to 218%. In terms of consumption, the welfare loss rises from 38% to 67%.

4.4 Ability Distribution

Subsection 3.2 showed that the nonlinear relationship between inflation and growth requires a long and fat-tailed distribution (see Proposition 3). A sufficiently large h_{max} implies a long-tailed distribution. Panel (a) of Figure 3 examines the effects of h_{max} on the inflation–growth relationship. This panel uses the values of h_{max} below the benchmark (h_{max} = benchmark /100 and benchmark /10, 000). If $a \in [1, 2)$, the ability distribution has a fat tail ($\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max}) = +\infty$). The benchmark calibration sets $a = 1$. Panel (b) of Figure 3 presents the inflation–growth relationships for $a = 1.4$ and 1.8.[12] Both panels plot the homogeneous cases with benchmark parameters for comparison. As the value of h_{max} becomes smaller and as the value of a becomes closer to 2, the nonlinearity becomes weaker. However, the heterogeneous case still creates the stronger nonlinearity between inflation and growth than the homogeneous case. Heterogeneous ability may thus be a source of nonlinearity for a wide range of ability distribution.

[Figure 3]

4.5 CIA Constraints

Our model needs CIA constraints on the variable and fixed costs of intermediate good production to ensure that the nominal interest rate has a growth effect.[13] Therefore, the CIA constraints on these costs are essential for generating the

[12] In the heterogeneous cases of Figure 3, we adjust the value of δ for each value of h_{max} and a to ensure that the growth rate is 2% when the inflation rate is 2%.

[13] Remember that Assumption 2 is needed for the nominal interest rate to have a growth effect. On the contrary, the CIA constraint on consumption, represented by θ_c, does not influence our results because it does not affect the threshold ability.

nonlinear relationship between inflation and growth. The nonlinear relationship in our model may be sensitive to the values of θ_η and θ_ξ. However, the benchmark analysis so far sets the values of the strength of the CIA constraints arbitrarily, $\theta_\eta = \theta_\xi = 1$. Keeping the values of the other parameters unchanged, this subsection changes the values of θ_η and θ_ξ and examines their effects.

Figure 4 shows the relationships between inflation and growth for the different values of θ_η and θ_ξ. Even if we use the smaller values of θ_η and θ_ξ, which means looser CAI constraints, the inflation–growth relationship is highly nonlinear in the presence of heterogeneity in ability, while it is roughly linear without heterogeneity in ability.

[Figure 4]

Moreover, Figure 4 reveals an interesting point. When the CIA constraints become looser, the growth rate increases irrespective of the presence of heterogeneity. This is because loosening the CAI constraints on the variable and fixed costs increases the net profit of intermediate good firms, $\widehat{\pi}$, which has a positive effect on the benefit of being an entrepreneur. More importantly, the presence of heterogeneity affects the magnitude of the increases in the growth rate. Without heterogeneity, loosening the CIA constraint stimulates growth significantly for all inflation rates. However, in the heterogeneous ability economy, the magnitude of the increases in the growth rate varies substantially depending on the inflation rate. For low inflation, the growth effect of loosening the CIA constraint is limited. For high inflation, a looser CAI constraint accelerates growth significantly. This result suggests that financial development may have only a limited growth effect in low-inflation countries, while it may have a substantial growth effect in high-inflation countries.

5 Conclusion

In this study, we derived an empirically plausible nonlinear relationship between inflation and growth both analytically and numerically. By using an R&D-based endogenous growth model with money, we showed that the nonlinearity of the inflation–growth relation depends on three key factors: (i) the CIA constraint on intermediate good production, (ii) heterogeneity in entrepreneurial ability, and (iii) a "long and fat-tailed" distribution of ability.

In our model, owing to the presence of the CIA constraint on intermediate good production, a higher inflation rate depresses the monopolistic profits earned by intermediate good firms. This in turn reduces the number of entrepreneurs, which lowers economic growth. When the inflation rate is high, only high-ability agents become entrepreneurs. When the distribution of ability has a "long- and fat tail," there is a nonnegligible number of high-ability agents. Therefore, the occupational choices of these high-ability agents have a large impact on growth, which results in a strong negative relationship between inflation and growth.

We also present numerical examples to examine whether the nonlinearity between inflation and growth is obtained under plausible parameter values. By using parameters consistent with U.S. data, we show that for an inflation rate above 20%, our model economy exhibits a highly nonlinear relationship between inflation and growth. This result is consistent with the empirical evidence.

Appendix

A An Alternative Setting

This appendix modifies our basic model so that individual agents are the decision makers. Here, we retain the notation used in the main text as far as possible. The utility of an agent with ability h at time s is given by

$$U_{h,s} = \int_s^\infty \frac{(c_{h,t})^{1-\sigma} - 1}{1-\sigma} \cdot e^{-\rho(t-s)} \mathrm{d}t, \tag{A.1}$$

where $c_{h,t}$ denotes the final good consumption of an agent with ability h at time t.

We denote the number of intermediate firms owned by an agent with ability h at time t by $n_{h,t}$. We have $N_t = \int n_{h,t} L dF(h)$. The law of motion for $n_{h,t}$ is given by

$$\dot{n}_{h,t} = \delta K_t h \cdot (1 - I_{h,t}). \tag{A.2}$$

Again, the operating profit of intermediate good j is given by (6). The flow

budget constraint of an agent with ability h is

$$c_{h,t} + \dot{b}_{h,t} + \dot{m}_{h,t} = r_t b_{h,t} - i_{R,t} x_{h,t} + I_{h,t} w_t + \int_0^{n_{h,t}} \pi_{i,t} \mathrm{d}i + \frac{T_t}{L} - \mu_t m_{h,t}, \quad \text{(A.3)}$$

where $b_{h,t}$ and $m_{h,t}$ denote the agent's real bond and real money holdings, respectively. Agents can borrow real money from other agents by incurring the money rental rate $i_{R,t}$. In (A.3), $x_{h,t}$ denotes the real money borrowed from other agents. A negative $x_{h,t}$ means that the agent with ability h lends real money to other households. Since agents cannot lend real money beyond their real money holdings, $m_{h,t} + x_{h,t} \geq 0$ must hold. Naturally, we have $\int x_{h,t} L \mathrm{d}F(h) = 0$. Now, the CIA constraint is given by

$$m_{h,t} + x_{h,t} \geq \theta_c c_{h,t} + \theta_\eta \int_0^{n_{h,t}} \eta z_{j,t} \mathrm{d}j + \theta_\xi n_{h,t} \xi. \quad \text{(A.4)}$$

Given $b_{h,0}$, $m_{h,0}$, and $n_{h,0}$, an agent with ability h maximizes (A.1) subject to (6), (A.2), (A.3), and (A.4). The first-order conditions are given by

$$c_{h,t} \ : \ (c_{h,t})^{-\sigma} = \lambda_{h,t} + \theta_c \psi_{h,t}, \quad \text{(A.5)}$$

$$x_{h,t} \ : \ \lambda_{h,t} i_{R,t} = \psi_{h,t}, \quad \text{(A.6)}$$

$$z_{i,t} \ : \ \lambda_{h,t} \left\{ (1-\alpha)^2 l_t^\alpha z_{i,t}^{-\alpha} - \eta \right\} = \psi_{h,t} \theta_\eta \eta, \quad \text{(A.7)}$$

$$b_{h,t} \ : \ \dot{\lambda}_{h,t} = (\rho - r_t) \lambda_{h,t}, \quad \text{(A.8)}$$

$$m_{h,t} \ : \ -\lambda_{h,t} \mu_t + \psi_{h,t} = -\dot{\lambda}_{h,t} + \rho \lambda_{h,t}, \quad \text{(A.9)}$$

$$n_{h,t} \ : \ \lambda_{h,t} \cdot \left\{ (1-\alpha) l_t^\alpha z_{n_{h,t},t}^{1-\alpha} - \eta z_{n_{h,t},t} - \xi \right\} - \psi_{h,t} \cdot (\theta_\eta \eta z_{n_{h,t},t} + \theta_\xi \xi) = -\dot{\zeta}_{h,t} + \rho \zeta_{h,t}, \quad \text{(A.10)}$$

$$I_{h,t} \ : \ I_{h,t} = \begin{cases} 1 & \text{if } \lambda_{h,t} w_t > \zeta_{h,t} \delta N_t h, \\ 0 & \text{if } \lambda_{h,t} w_t \leq \zeta_{h,t} \delta K_t h, \end{cases} \quad \text{(A.11)}$$

where $\lambda_{h,t}$, $\zeta_{h,t}$, and $\psi_{h,t}$ are the costate variables associated with the budget constraint, law of motion for $n_{h,t}$, and CIA constraint, respectively.

From (A.6), (A.8), (A.9), and the Fisher equation, we obtain

$$i_{R,t} = r_t + \mu_t \equiv i > 0. \quad \text{(A.12)}$$

The money rental rate becomes equal to the nominal interest rate. (A.6) and (A.12), together with $c_{h,t} > 0$, imply $\psi_{h,t} > 0$ for all h and $t \geq 0$. Then, the CIA

constraints of all agents are always binding, which ensures $m_{h,t} + x_{h,t} = 0$. From (A.5), (A.6), and (A.8), we obtain the following consumption Euler equation:

$$\frac{\dot{c}_{h,t}}{c_{h,t}} = \frac{1}{\sigma} \cdot (r_t - \rho). \tag{A.13}$$

From (A.6) and (A.7), we know that z_t is still given by (12). By using (A.6), we rewrite (A.10) as $\lambda_{h,t} \cdot \widehat{\pi}_t = -\dot{\zeta}_{h,t} + \rho\zeta_{h,t}$, where $\widehat{\pi}_t \equiv (1-\alpha)l_t^{\alpha} z_t^{1-\alpha} - (1+i\theta_\eta)\eta z_t - (1 + i\theta_\xi)\xi$. Let us define $\nu_{h,t} = \zeta_{h,t}/\lambda_{h,t}$. Then, we have $r_t \nu_{h,t} = \dot{\nu}_{h,t} + \widehat{\pi}_t$, which has the following solution: $\nu_{h,t} = \int_t^{\infty} \widehat{\pi}_\tau e^{-\int_t^\tau r_u du} \mathrm{d}\tau \equiv \nu_t$. Thus, $\nu_{h,t}$ represents the value of an intermediate good firm and is independent of ability h. From (A.11), h_t^* satisfies (14) again. We obtain exactly the same conditions as in our basic model.

B The Dynamics of h_t^*

Recall that $l_t = F(h_t^*)L$. Inserting (1) and (12) into (16c) yields

$$F(h_t^*)LN_t \cdot \left[\frac{(1-\alpha)^2}{\eta(1+i\theta_\eta)}\right]^{\frac{1-\alpha}{\alpha}} = Lc_t + N_t \eta F(h_t^*)L \cdot \left[\frac{(1-\alpha)^2}{\eta(1+i\theta_\eta)}\right]^{\frac{1}{\alpha}} + N_t\xi. \tag{B.1}$$

Define $\widehat{c_t} \equiv c_t/N_t$. The above equation can be rewritten as

$$\widehat{c_t} = F(h_t^*) \cdot \left[\frac{(1-\alpha)^2}{\eta(1+i\theta_\eta)}\right]^{\frac{1-\alpha}{\alpha}} \cdot \left[1 - \frac{(1-\alpha)^2}{1+i\theta_\eta}\right] - \frac{\xi}{L}. \tag{B.2}$$

Differentiating (B.2) with respect to time yields

$$\frac{\dot{\widehat{c_t}}}{\widehat{c_t}} = \Omega(h_t^*; i) \cdot \frac{\dot{h}_t^*}{h_t^*}, \tag{B.3}$$

where

$$\Omega(h_t^*; i) \equiv \frac{1}{\widehat{c_t}} \cdot \left[\frac{(1-\alpha)^2}{\eta(1+i\theta_\eta)}\right]^{\frac{1-\alpha}{\alpha}} \cdot \left[1 - \frac{(1-\alpha)^2}{1+i\theta_\eta}\right] \cdot F'(h_t^*)h_t^*.$$

From (11), (15), and the definition of $\widehat{c_t}$, we obtain

$$\frac{\dot{\widehat{c_t}}}{\widehat{c_t}} = \frac{\dot{c}_t}{c_t} - g(h_t^*) = \frac{1}{\sigma} \cdot (r_t - \rho) - g(h_t^*), \tag{B.4}$$

where $g(h_t^*) \equiv \delta LH(h_t^*)$ is the growth rate of N_t.

By using (12) and $l_t = LF(h_t^*)$, we rewrite $\widehat{\pi}_t$ as

$$\widehat{\pi}_t = \Pi \left\{(1 + i\theta_\eta)\eta\right\}^{-\frac{1-\alpha}{\alpha}} F(h_t^*) - (1 + i\theta_\xi)\xi, \tag{B.5}$$

where $\Pi \equiv \alpha(1-\alpha)^{\frac{2-\alpha}{\alpha}} L$. Substituting (12) into (3) yields

$$w_t = \alpha(1-\alpha)^{\frac{2(1-\alpha)}{\alpha}} \eta^{-\frac{1-\alpha}{\alpha}} \cdot (1 + i\theta_\eta)^{-\frac{1-\alpha}{\alpha}} \cdot N_t. \tag{B.6}$$

By using (B.6), we can rewrite (14) as

$$\alpha(1-\alpha)^{\frac{2(1-\alpha)}{\alpha}} \left\{(1 + i\theta_\eta)\eta\right\}^{-\frac{1-\alpha}{\alpha}} = \nu_t \delta h_t^*. \tag{B.7}$$

Since the nominal interest rate i is constant, (B.7) implies

$$\frac{\dot{h}_t^*}{h_t^*} = -\frac{\dot{\nu}_t}{\nu_t}. \tag{B.8}$$

Therefore, from (9c), (B.5), (B.8), and $r_t \nu_{h,t} = \dot{\nu}_{h,t} + \widehat{\pi}_t$, we have

$$\frac{\dot{h}_t^*}{h_t^*} = \frac{\Pi \left\{(1 + i\theta_\eta)\eta\right\}^{-\frac{1-\alpha}{\alpha}} F(h_t^*) - (1 + i\theta_\xi)\xi}{\nu_t} - r_t. \tag{B.9}$$

If we use (B.7), the above equation can be written as

$$\frac{\dot{h}_t^*}{h_t^*} = \frac{h_t^* \left\{\Pi F(h_t^*) - \Phi(i)\right\}}{\Gamma} - r_t, \tag{B.10}$$

where $\Gamma \equiv (1/\delta) \cdot \alpha(1-\alpha)^{\frac{2(1-\alpha)}{\alpha}}$ and $\Phi(i) \equiv \left\{(1 + i\theta_\eta)\eta\right\}^{\frac{1-\alpha}{\alpha}} (1 + i\theta_\xi)\xi$.

Finally, let us eliminate r_t from (B.10). From (B.3), (B.4), and (B.10), we have

$$\begin{aligned} r_t &= r(h_t^*; i) \\ &= \frac{1}{1 + \sigma\Omega(h_t^*; i)} \cdot \left\{\rho + \sigma g(h_t^*) + \frac{\sigma h_t^* \left\{\Pi F(h_t^*) - \Phi(i)\right\} \Omega(h_t^*; i)}{\Gamma}\right\} \end{aligned} \tag{B.11}$$

By inserting (B.11) into (B.10), we obtain (17).

C Proof of Proposition 1 and the Effect of i

For any $i \in (0, i_{max})$, let us define $\underline{h}(i)$ by $\Pi \cdot F(\underline{h}(i)) = \Phi(i)$. Note that $\underline{h}(i) > h_{min}$ because of $F(\underline{h}(i)) = \Phi(i)/\Pi > 0$. Equation (17) implies

$$\operatorname{sign}\frac{\dot{h}_t^*}{h_t^*} = \operatorname{sign}\left[\Psi(h_t^*; i) - g(h_t^*)\right], \tag{C.1}$$

$$\text{where} \quad \Psi(h_t^*; i) \equiv \frac{1}{\sigma} \cdot \left\{ \frac{h_t^* \left\{\Pi F(h_t^*) - \Phi(i)\right\}}{\Gamma} - \rho \right\}.$$

The definition of $\underline{h}(i)$ implies $\Psi(\underline{h}(i); i) = -\rho/\sigma < 0$ for any $i \in (0, i_{max})$. Since $\Phi(i)$ is an increasing function of i, Assumption 1 ensures $\Psi(h_{max}; i) > 0$ for any $i \in (0, i_{max})$. Moreover, for $h_t^* \in (\underline{h}(i), h_{max})$, we have $\mathrm{d}\Psi(h_t^*; i)/\mathrm{d}h_t^* > 0$, where $i \in (0, i_{max})$.

From $g(h_t^*) \equiv \delta L H(h_t^*)$, we obtain $g(h_{min}) > 0$, $g'(h_t^*) < 0$, and $g(h_{max}) = 0$. Figure 5 shows the graphs of $\Psi(h_t^*; i)$ and $g(h_t^*)$. There exists a unique steady-state equilibrium where h_t^* is constant at $h^*(i)$. Since the steady state is unstable, the economy is always in the steady-state equilibrium.

[Figures 5 and 6 about here.]

In the steady-state equilibrium, $\widehat{c_t}$ becomes constant (see (B.2)). Then, c_t grows at the same rate as N_t. Because of $l_t = LF(h^*(i))$ and (12), z_t becomes constant. Then, (16c) shows that Y_t also grows at the same rate as N_t. (16b) implies $\dot{M}_t/M_t = g(i) + \mu_t$. Finally, an increase in i shifts $\Psi(h_t^*; i)$ downward. Hence, $h^*(i)$ is an increasing function of i with $h^*(i_{max}) = h_{max}$ (see Figure 6). Thus, we have $g(i) > 0$ if $0 < i < i_{max}$ and $g(i_{max}) = 0$.

D Derivation of Equation (21)

From (18a), we obtain

$$\frac{\mathrm{d}g(i)}{\mathrm{d}i} = -\delta L h^*(i) F'(h^*(i)) \cdot \frac{\mathrm{d}h^*(i)}{\mathrm{d}i}. \tag{D.1}$$

From (C.1), we have $g(i) = \Psi(h^*(i); i)$ in the steady-state equilibrium. Then, we obtain

$$\frac{\mathrm{d}g(i)}{\mathrm{d}i} = \frac{\{\Pi F(h^*(i)) - \Phi(i)\} + h^*(i)\Pi F'(h^*(i))}{\sigma\Gamma} \cdot \frac{\mathrm{d}h^*(i)}{\mathrm{d}i} - \frac{h^*(i)\Phi'(i)}{\sigma\Gamma}. \tag{D.2}$$

By inserting (D.1) into (D.2), we obtain

$$\frac{\mathrm{d}g(i)}{\mathrm{d}i} = -\frac{\delta L h^*(i)\Phi'(i)}{\Pi + \sigma\Gamma\delta L + \dfrac{\Pi F(h^*(i)) - \Phi(i)}{h^*(i)F'(h^*(i))}}. \tag{D.3}$$

Finally, $g(i) = \Psi(h^*(i); i)$ implies

$$\Pi F(h^*(i)) - \Phi(i) = \frac{\Gamma(\rho + \sigma g(i))}{h^*(i)}. \tag{D.4}$$

By inserting (D.4) into (D.3), we obtain (21).

E A Homogeneous Ability Economy

Although most of the first-order conditions in a heterogeneous ability economy can be applied to the homogeneous ability case, (9f) must be modified. We retain the notation used in our heterogeneous ability model as far as possible. We consider a steady-state equilibrium where q_t is constant. In an equilibrium where there are both workers and entrepreneurs ($q \in (0,1)$), (9f) is replaced by $\lambda_t w_t = \zeta_t \delta N_t \widehat{h}$. Accordingly, (14) is replaced by $w_t = \nu_t \cdot \delta N_t \widehat{h}$.

In a homogeneous ability economy, ν_t and w_t are still given by (13) and (B.6), respectively. In a steady-state equilibrium, (13) implies $\nu = \widehat{\pi}/r$. The Euler equation (11) implies $r = \rho + \sigma g^H$. Since the number of workers is $l = Lq$, $\widehat{\pi}_t$ is now given by

$$\widehat{\pi}_t = \Pi\{\eta(1 + i\theta_\eta)\}^{-\frac{1-\alpha}{\alpha}} q - (1 + i\theta_\xi)\xi. \tag{E.1}$$

By using (B.6), (E.1), $\nu = \widehat{\pi}/r$, and $r = \rho + \sigma g^H$, we can rewrite $w_t = \nu_t \cdot \delta N_t \widehat{h}$ as

$$\Gamma = \widehat{h}\frac{\Pi q - \Phi(i)}{\rho + \sigma g^H}, \quad \text{or} \quad g^H = \frac{1}{\sigma} \cdot \left\{ \frac{\widehat{h}\{\Pi q - \Phi(i)\}}{\Gamma} - \rho \right\}. \tag{E.2}$$

From this equation together with $g^H = \delta \widehat{h} L(1 - q)$, we obtain (23) and

$$q(i) = \frac{\widehat{h}(\Phi(i) + \sigma\Gamma\delta L) + \rho\Gamma}{\widehat{h}(\Pi + \sigma\Gamma\delta L)}. \tag{E.3}$$

F Proof of Proposition 2

From (E.2), $g(i) = \Psi(h^*(i); i)$ and $g^H(\bar{i}) = g(\bar{i}) > 0$, we obtain

$$g^H(\bar{i}) = \frac{1}{\sigma} \cdot \left\{ \frac{\widehat{h}\left\{\Pi q(\bar{i}) - \Phi(\bar{i})\right\}}{\Gamma} - \rho \right\} = \frac{1}{\sigma} \cdot \left\{ \frac{h^*(\bar{i})\left\{\Pi F(h^*(\bar{i})) - \Phi(\bar{i})\right\}}{\Gamma} - \rho \right\} = g(\bar{i}) > 0,$$

which implies

$$\Pi \cdot (\widehat{h}q(\bar{i}) - h^*(\bar{i})F(h^*(\bar{i}))) = \Phi(\bar{i}) \cdot (\widehat{h} - h^*(\bar{i})). \tag{F.1}$$

From the above equation together with $\Pi > 0$ and $\Phi(\bar{i}) > 0$, the following holds:

$$\text{sign}\left\{\widehat{h}q(\bar{i}) - h^*(\bar{i})F(h^*(\bar{i}))\right\} = \text{sign}\{\widehat{h} - h^*(\bar{i})\}. \tag{F.2}$$

From (E.2) and $g^H(\bar{i}) > 0$, $\Pi q(\bar{i}) > \Phi(\bar{i})$ holds, which implies $\Pi > \Phi(\bar{i})$ because of $q(\bar{i}) \in (0, 1)$. Then, (F.1) implies

$$|\widehat{h}q(\bar{i}) - h^*(\bar{i})F(h^*(\bar{i}))| < |\widehat{h} - h^*(\bar{i})|. \tag{F.3}$$

From the definitions of $g^H(i)$ and $g(i)$, we have

$$g^H(\bar{i}) = \delta L\widehat{h}(1 - q(\bar{i})), \tag{F.4}$$

$$g(\bar{i}) = \delta L \int_{h^*(\bar{i})}^{h_{max}} h\mathrm{d}F(h) > \delta L \int_{h^*(\bar{i})}^{h_{max}} h^*(\bar{i})\mathrm{d}F(h) = \delta L h^*(\bar{i})\{1 - F(h^*(\bar{i}))\}. \tag{F.5}$$

Since $g^H(\bar{i}) = g(\bar{i})$, the above two equations imply

$$\delta L\widehat{h}(1 - q(\bar{i})) > \delta L h^*(\bar{i})\{1 - F(h^*(\bar{i}))\} \;\Rightarrow\; \widehat{h} - h^*(\bar{i}) > \widehat{h}q - h^*(\bar{i})F(h^*(\bar{i})). \tag{F.6}$$

Now, we prove $h^*(\bar{i}) < \widehat{h}$ by contradiction. Assume that $h^*(\bar{i}) = \widehat{h}$. Then, (F.6) implies $0 = \widehat{h} - h^*(\bar{i}) > \widehat{h}q(\bar{i}) - h^*(\bar{i})F(h^*(\bar{i}))$, which contradicts equation (F.2). Next, assume that $h^*(\bar{i}) > \widehat{h}$. Then, (F.6) implies $0 > \widehat{h} - h^*(\bar{i}) > \widehat{h}q - h^*(\bar{i})F(h^*(\bar{i}))$, which contradicts (F.3). Therefore, we can conclude that $\widehat{h} > h^*(\bar{i})$.

Since $h^*(i)$ increases with i, $h^*(i) < \widehat{h}$ holds for $i \leq \bar{i}$. Then, for any $i \leq \bar{i}$, we

have

$$\begin{aligned} 0 > \Delta_{i,g}(i) &= -\frac{\delta L h^*(i)\Phi'(i)}{\Pi + \sigma\Gamma\delta L + \dfrac{\Gamma(\rho+\sigma g(i))}{(h^*(i))^2 F'(h^*(i))}} \\ &> -\frac{\delta L h^*(i)\Phi'(i)}{\Pi + \sigma\Gamma\delta L} \quad \left(\because \frac{\Gamma(\rho+\sigma g(i))}{(h^*(i))^2 F'(h^*(i))} > 0\right) \\ &> -\frac{\delta L \widehat{h}\Phi'(i)}{\Pi + \sigma\Gamma\delta L} \quad \left(\because \widehat{h} > h^*(i)\right) \\ &= \Delta^H_{i,g}(i), \end{aligned}$$

where $\Delta_{i,g}(i) \equiv \mathrm{d}g(i)/\mathrm{d}i$ and $\Delta^H_{i,g}(i) \equiv \mathrm{d}g^H(i)/\mathrm{d}i$. Since $\Delta_{i,g}(i) > \Delta^H_{i,g}(i)$ holds at $i = \overline{i}, \overline{i}$ must be unique from the continuity of $g(i)$ and $g^H(i)$.

Remember that a large $|\Delta_{i,g}(i)|$ ($|\Delta^H_{i,g}(i)|$) implies a large $|\Delta_{\mu,g}(i)|$ ($|\Delta^H_{\mu,g}(i)|$) (see (22)). Then, for any $i \in (0, \overline{i}]$, we have $0 > \Delta_{\mu,g}(i) > \Delta^H_{\mu,g}(i)$.

G Proof of Proposition 3

First, let us consider the homogeneous ability case. From (18b), we have $0 < \Phi'(i) < +\infty$ for all $i \in (0, i^H_{max})$. Thus, (24) implies $0 > \Delta^H_{i,g}(i)(\equiv \frac{\mathrm{d}g^H(i)}{\mathrm{d}i}) > -\infty$ for all $i \in (0, i^H_{max})$. From (22), we have $0 > \Delta^H_{\mu,g}(i) > -\infty$ for all $i \in (0, i^H_{max})$.

We next consider the heterogeneous ability case. As shown in Figure 6, we have $h^*(i_{max}) = h_{max}$ and $g(i_{max}) = 0$. Then, (21) can be written as

$$\Delta_{i,g}(i_{max}) \equiv \left.\frac{\mathrm{d}g(i)}{\mathrm{d}i}\right|_{i=i_{max}} = -\frac{\delta L \Phi'(i_{max})}{\dfrac{\Pi + \sigma\Gamma\delta L}{h_{max}} + \dfrac{\Gamma\rho}{(h_{max})^3 F'(h_{max})}}.$$

Let us take a limit $h_{max} \to +\infty$. From (19), we have $\lim_{h_{max}\to+\infty} i_{max} = \overline{i_{max}}$, where $\overline{i_{max}}$ satisfies $\Pi = \Phi(\overline{i_{max}})$. Then, we obtain

$$\lim_{h_{max}\to+\infty} \Delta_{i,g}(i_{max}) = \lim_{h_{max}\to+\infty} \left.\frac{\mathrm{d}g(i)}{\mathrm{d}i}\right|_{i=i_{max}} = -\frac{\delta L \Phi'(\overline{i_{max}})}{\dfrac{\Gamma\rho}{\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max})}}.$$

Since $\Phi'(\overline{i_{max}})$ is finite, when $\lim_{h_{max}\to+\infty}(h_{max})^3 F'(h_{max}) = +\infty$, we obtain

$$\lim_{h_{max}\to+\infty} \Delta_{i,g}(i_{max}) = -\infty.$$

Therefore, when h_{max} is sufficiently large, we have

$$0 > \min_{i\in(0,i^H_{max})} \left\{\Delta^H_{i,g}(i)\right\} > \Delta_{i,g}(i_{max}).$$

Since $\Delta_{i,g}(i)(\equiv \mathrm{d}g(i)/\mathrm{d}i)$ is continuous in i, we have (27) for a sufficiently large $i \in (0, i_{max})$. Since a large $|\Delta_{i,g}(i)|$ implies a large $|\Delta_{\mu,g}(i)|$, (28) holds for a sufficiently large $i \in (0, i_{max})$.

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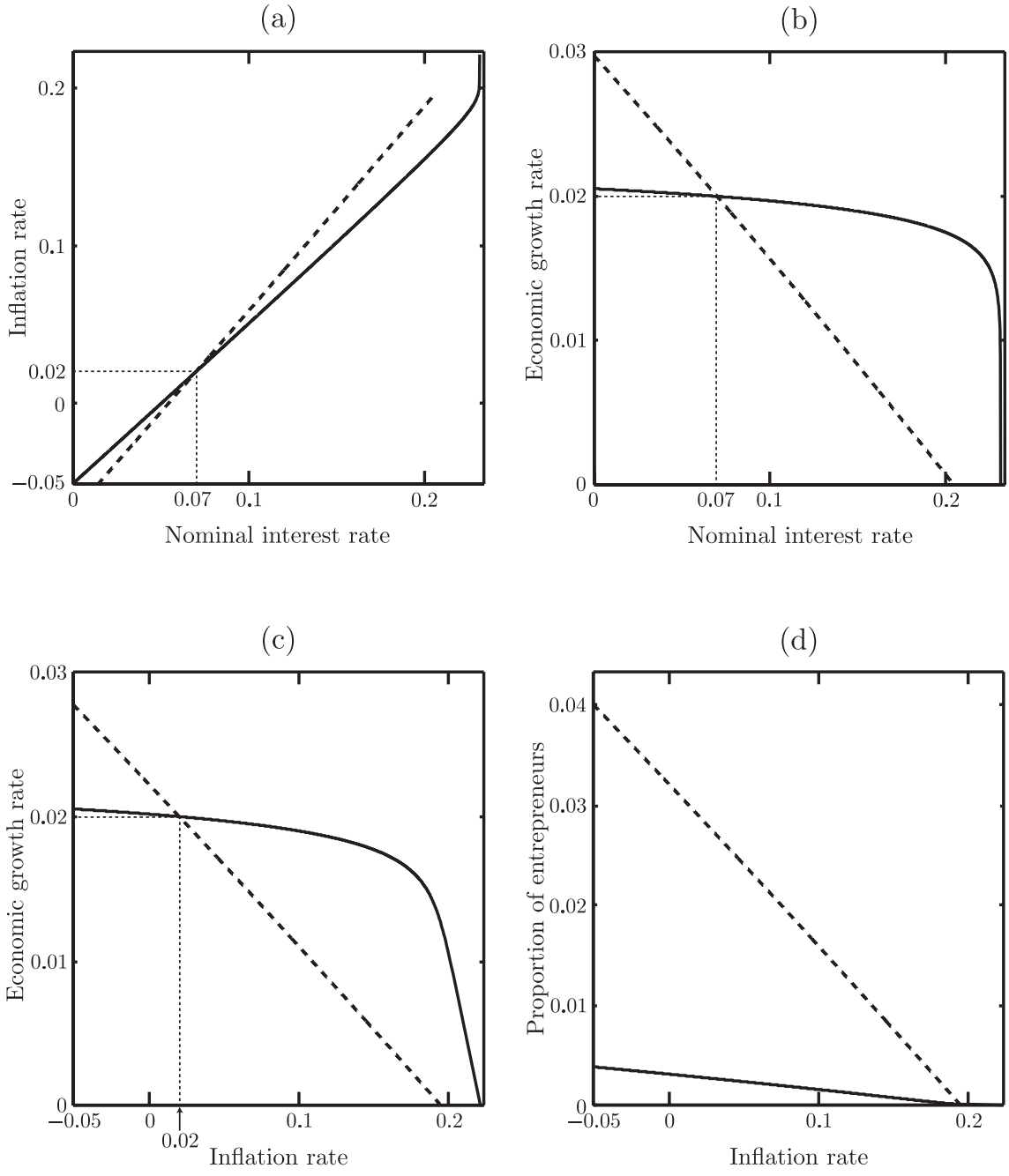

Figure 1: Nominal interest rate, inflation rate, and growth.
The solid lines show the graphs of the heterogeneous ability economy. The dashed lines show the graphs of the homogeneous ability economy.

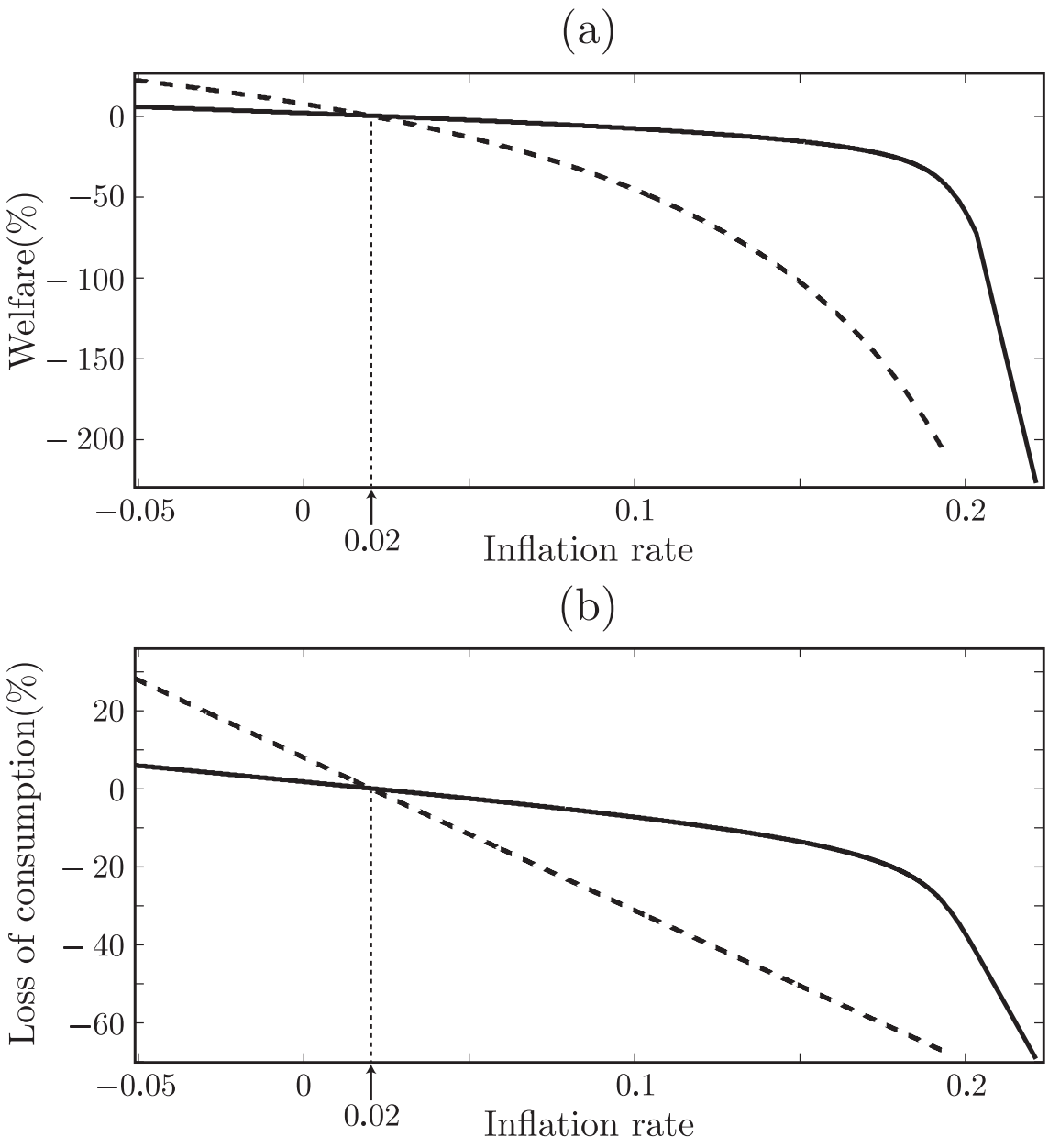

Figure 2: Inflation rate and welfare.
The solid lines show the graphs of the heterogeneous ability economy. The dashed lines show the graphs of the homogeneous ability economy.

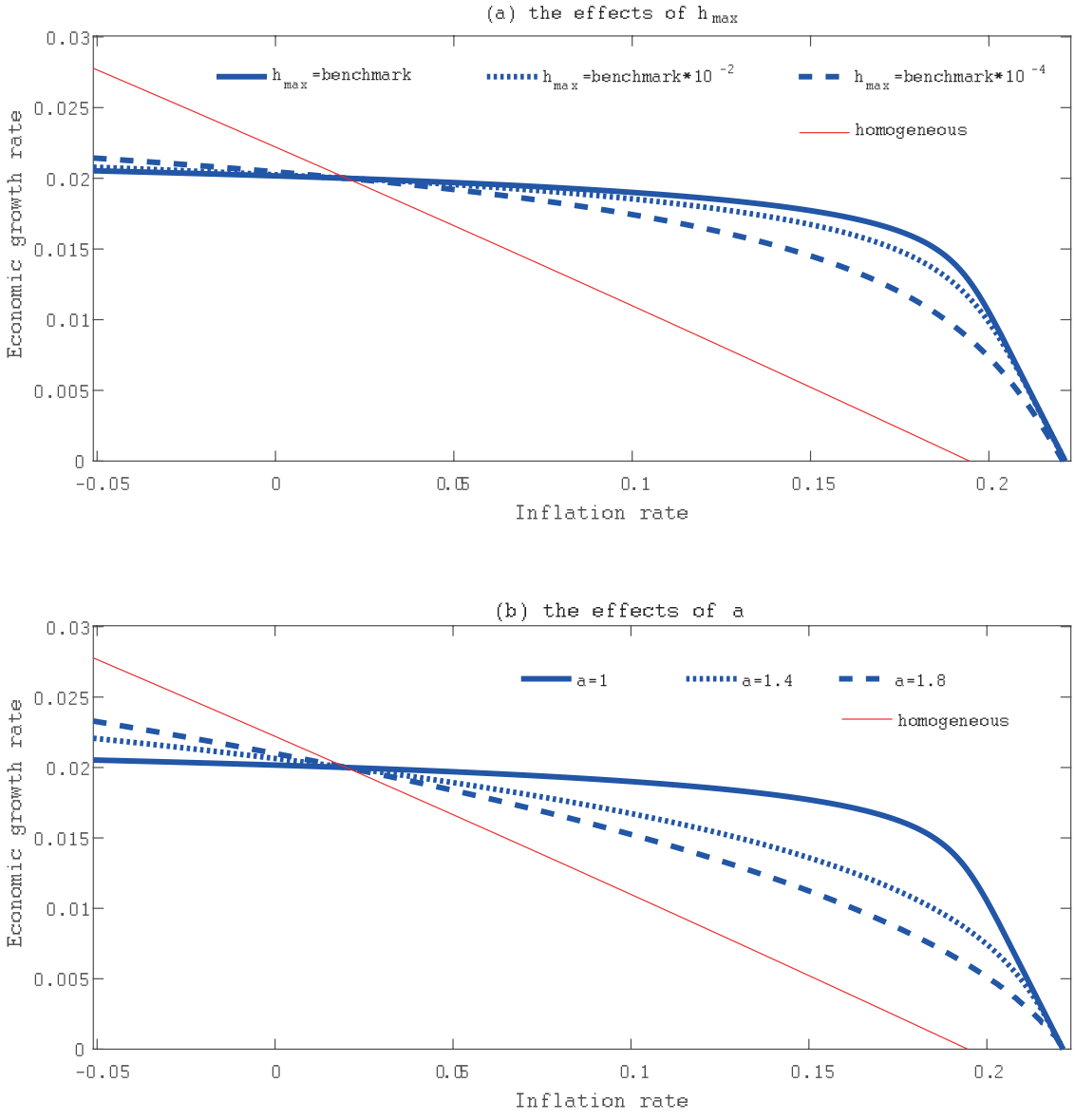

Figure 3: Ability distribution and nonlinearity.

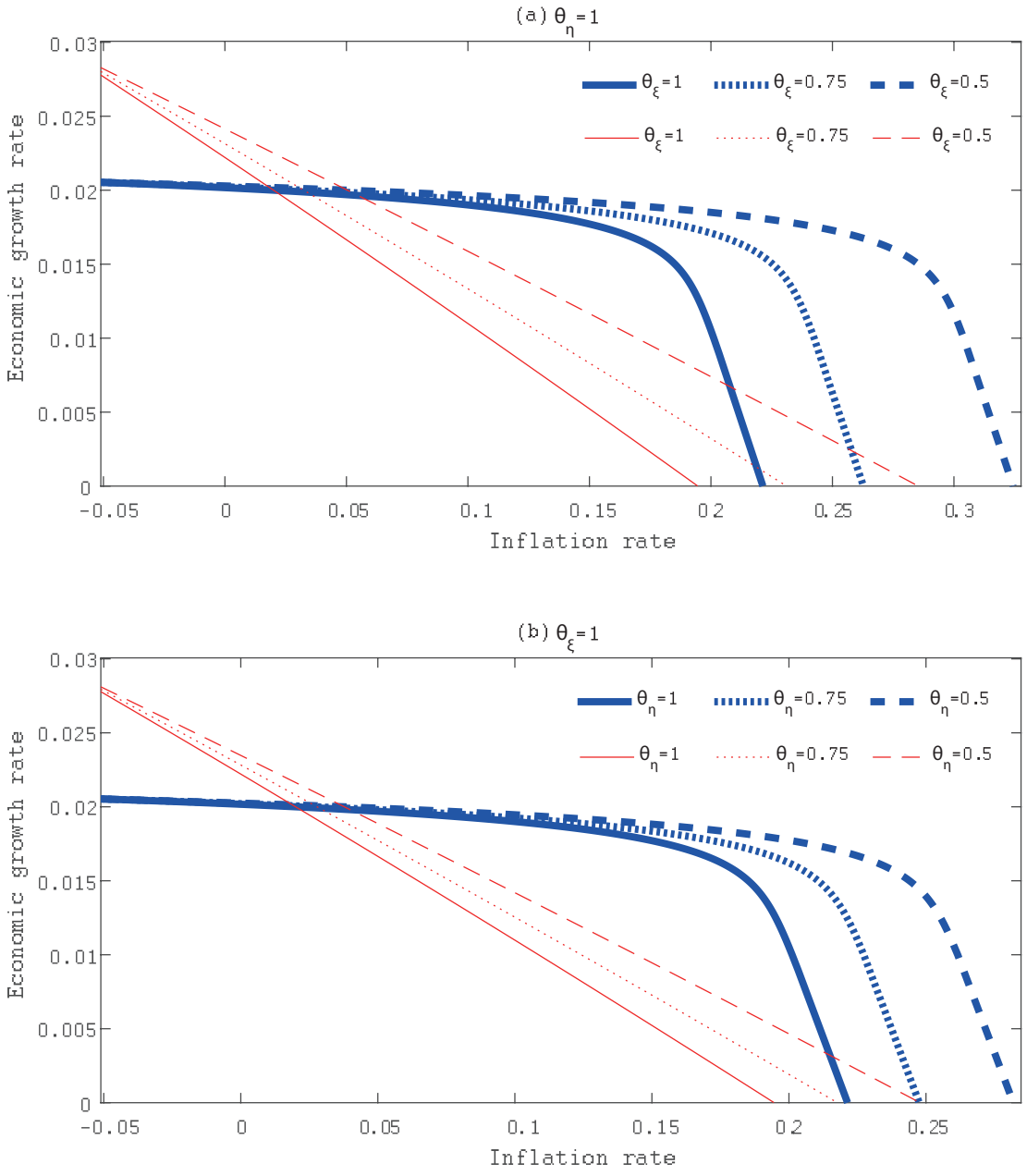

Figure 4: The effects of θ_η and θ_ξ.
The thick lines show the graphs of the heterogeneous ability economy. The thin lines show the graphs of the homogeneous ability economy.

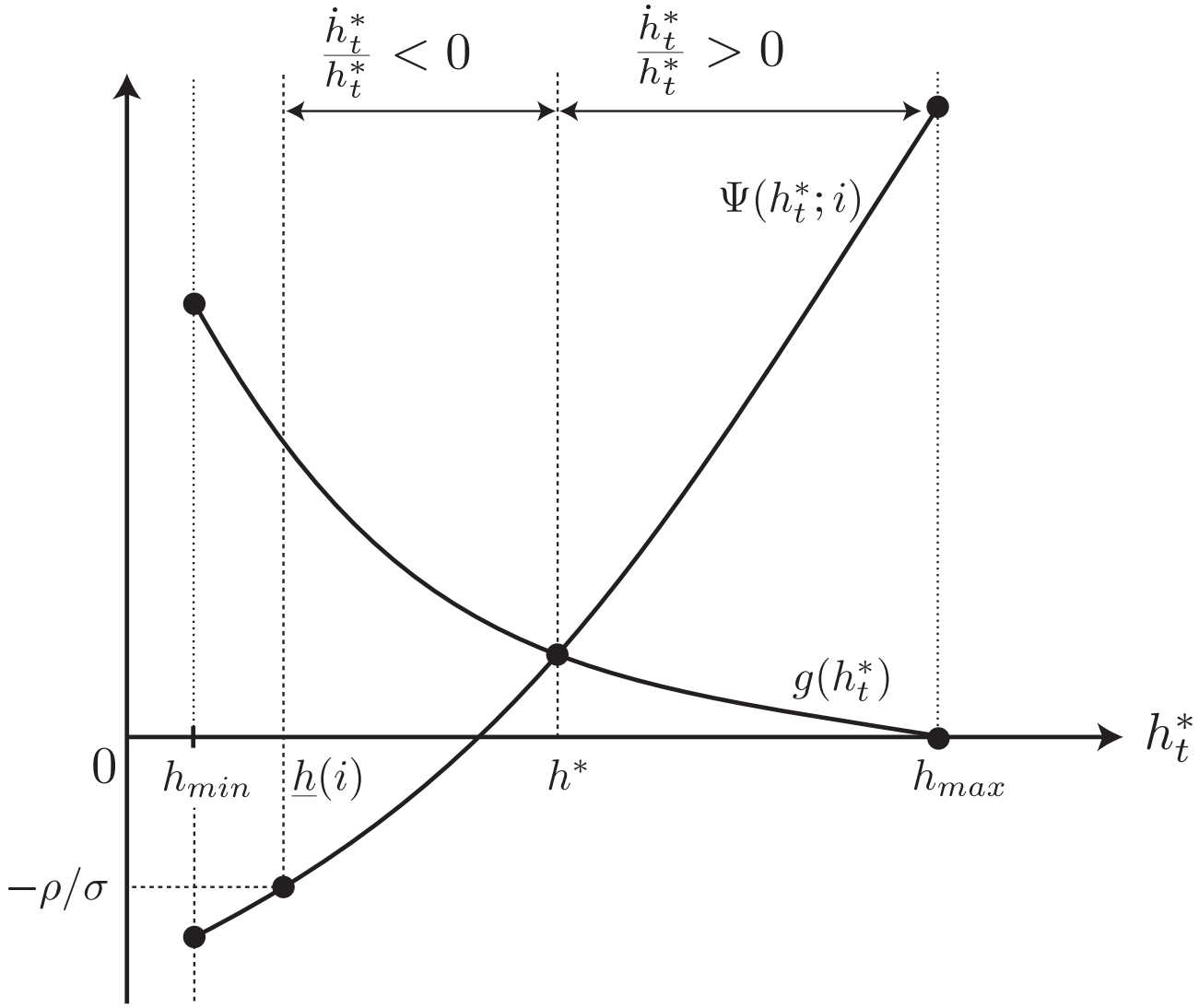

Figure 5: Steady State Equilibrium

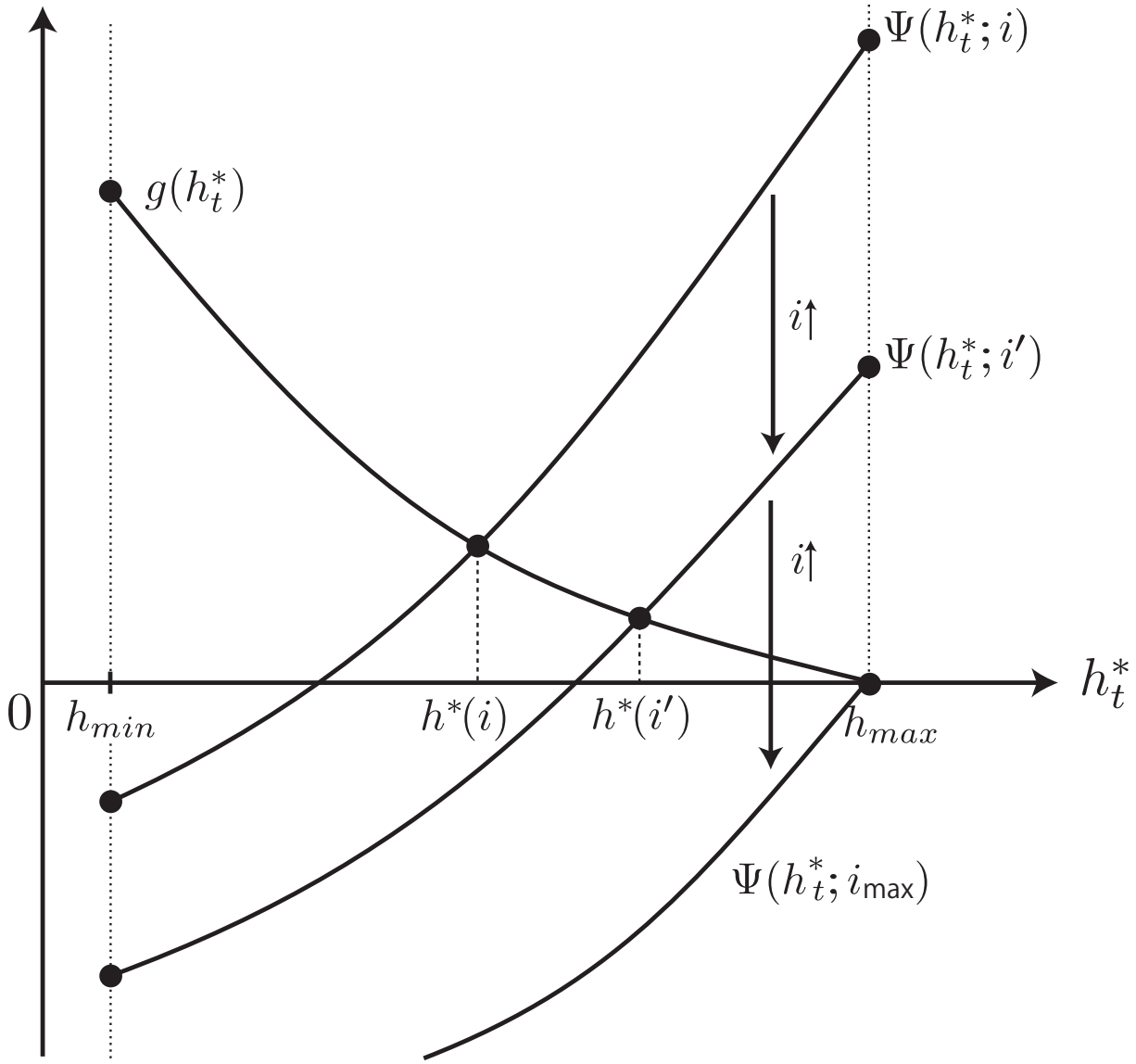

Figure 6: Effect of Nominal Interest Rate