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Abstract We analyze minimization algorithms, called the two-phase algorithms, for L$^\natural$-convex functions in discrete convex analysis and derive tight bounds for the number of iterations.

1 Introduction and Result

With motivations from auction theory, we discuss minimization of a discrete convex function called L$^\natural$-convex function. A function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ defined on integer lattice points is said to be *L$^\natural$-convex* [10] if for every $p, q \in \mathrm{dom}\, g$ and every nonnegative $\lambda \in \mathbb{Z}_+$, it holds that

$$g(p) + g(q) \geq g((p + \lambda \mathbf{1}) \wedge q) + g(p \vee (q - \lambda \mathbf{1})), \tag{1.1}$$

where $\mathrm{dom}\, g = \{p \in \mathbb{Z}^n \mid g(p) < +\infty\}$, $\mathbf{1} = (1, 1, \ldots, 1)$, and for $p, q \in \mathbb{Z}^n$ the vectors $p \wedge q$ and $p \vee q$ denote, respectively, the vectors of component-wise minimum and maximum of p and q. The concept of L$^\natural$-convex function plays a primary role in the theory of discrete convex analysis [10], and an important application can be found in auction theory, in addition to discrete optimization and computer vision (see [14]).

We consider a certain type of algorithm for L$^\natural$-convex function minimization, called the two-phase algorithm. While the two-phase algorithm and its variants are originally considered in [12, 13] for a specific L$^\natural$-convex function arising from an auction model (see Section 2 for details), the algorithms work for general L$^\natural$-convex functions.

As its name indicates, the two-phase algorithm consists of two phases, the up phase and the down phase. The algorithm starts from an arbitrarily chosen initial vector, and the vector moves upward in the up phase and then downward in the down phase. A detailed descrpition of the algorithm is as follows, where $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is an L$^\natural$-convex function, $N = \{1, 2, \ldots, n\}$, and $\chi_X \in \{0, 1\}^n$ is the characteristic vector of a set $X \subseteq N$.

Algorithm TwoPhase
  Step 0: Find a vector $p^\circ \in \mathrm{dom}\, g$ and set $p := p^\circ$. Go to Up Phase.
  Up Phase:
    Step U1: Find a minimizer $X \subseteq N$ of $g(p + \chi_X)$.
    Step U2: If $g(p + \chi_X) = g(p)$, then go to Down Phase.

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Step U3: Set $p := p + \chi_X$, and go to Step U1.
Down Phase:
Step D1: Find a minimizer $X \subseteq N$ of $g(p - \chi_X)$.
Step D2: If $g(p - \chi_X) = g(p)$, then output p and stop.
Step D3: Set $p := p - \chi_X$, and go to Step D1.

In this paper, we analyze the number of updates of vector p in the two-phase algorithm[1]. We denote

$$\mu(p^\circ) = \min\{\eta(p^*, p^\circ) \mid p^* \in \arg\min g\},$$

where $\arg\min g$ is the set of minimizers of g and

$$\begin{aligned} \eta(p,q) &= \|p-q\|_\infty^+ + \|p-q\|_\infty^- && (p, q \in \mathbb{Z}^n), \qquad (1.2)\\ \|q\|_\infty^+ &= \max(0, q(1), q(2), \ldots, q(n)) && (q \in \mathbb{Z}^n),\\ \|q\|_\infty^- &= \max(0, -q(1), -q(2), \ldots, -q(n)) && (q \in \mathbb{Z}^n). \end{aligned}$$

That is, $\mu(p^\circ)$ is the η-distance from vector p° to the nearest minimizer of g.

Theorem 1.1 *Suppose that the algorithm* TwoPhase *is applied to an* $L^\natural$*-convex function* g *with an initial vector* $p^\circ \in \mathrm{dom}\, g$*. Then, the number of updates of vector* p *is bounded by* $\mu(p^\circ)$ *in the up phase and by* $\mu(p^\circ)$ *in the down phase; in total, bounded by* $2\mu(p^\circ)$*.*

It is noted that while the minimizers X selected in Steps U1 and D1 are not uniquely determined, the bounds in Theorem 1.1 hold irrespectively of the choice of X.

The following numerical example shows that the bounds in the theorem above are tight.

Example 1.2 Let us consider a function $g : \mathbb{Z}^2 \to \mathbb{R} \cup \{+\infty\}$ given by

$$g(p(1), p(2)) = \begin{cases} -2(p(1) - p(2)) + \max\{0, p(1)\} & (\text{if } p(1) - p(2) \le k), \\ +\infty & (\text{otherwise}), \end{cases} \qquad (1.3)$$

where k is an arbitrary positive integer. It can be shown that g is an $\mathrm{L}^\natural$-convex function and

$$\arg\min g = \{(p(1), p(2)) \in \mathbb{Z}^2 \mid p(1) - p(2) = k,\ p(1) \le 0\}.$$

Suppose that the algorithm TwoPhase is applied to this function g with the initial vector $p^\circ = (0, 0)$. Then, the trajectory of vector p is as follows (see Figure 1):

$$\begin{aligned} &(0,0) \to (1,0) \to (2,0) \to \cdots \to (k,0)\\ &\qquad \to (k-1,-1) \to (k-2,-2) \to \cdots \to (0,-k). \end{aligned}$$

Therefore, the numbers of updates of vector p in the up and down phases are each equal to k. We also have $\mu(p^\circ) = \eta(p^*, p^\circ) = k$ with $p^* = (0, -k) \in \arg\min g$. Hence, we see that the bounds in Theorem 1.1 for the numbers of updates of vector p in the up and down phases are tight. □

We also consider a variant of TwoPhase in [13], named TwoPhaseMinMin, for finding a (unique) minimal minimizer of an $\mathrm{L}^\natural$-convex function[2]. The algorithm TwoPhaseMinMin is different from TwoPhase in the choice of X in each iteration and the termination condition of each phase. It is obtained from TwoPhase by changing Steps U1, U2, D1, and D2 to the following:

Step U1: Find the unique minimal minimizer $X \subseteq N$ of $g(p + \chi_X)$.
Step U2: If $X = \emptyset$, then go to Down Phase.
Step D1: Find the unique maximal minimizer $X \subseteq N$ of $g(p - \chi_X)$.
Step D2: If $X = \emptyset$, then output p and stop.

[1] A weaker statement than Theorem 1.1 is given in an unpublished technical report [12].
[2] Due to $\mathrm{L}^\natural$-convexity, a minimal minimizer is uniquely determined if it exists.

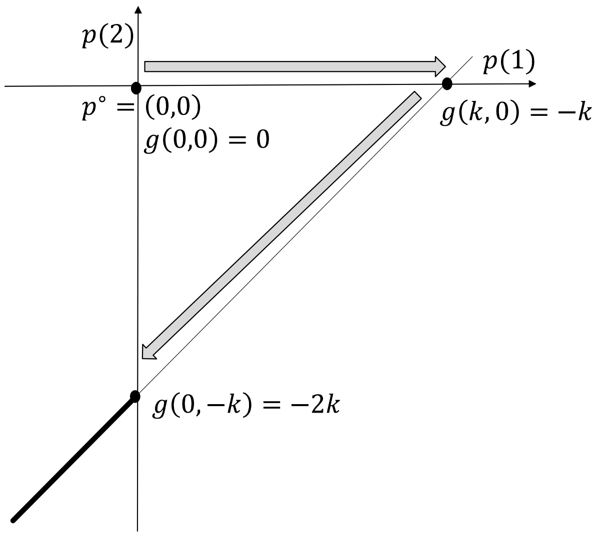

Fig. 1 Behavior of vector p in Algorithm TwoPhase applied to function g in (1.3) with $p^\circ = (0,0)$. The thick half-line corresponds to the set of minimizers of g.

The following bounds for the numbers of updates of p are shown in [13][3], where $p^*_{\min} \in \mathbb{Z}^n$ denotes the unique minimal minimizer of $\mathrm{L}^\natural$-convex function g.

Proposition 1.3 ([13, Theorem 4.13]) *Suppose that the algorithm* TwoPhaseMinMin *is applied to an $L^\natural$-convex function g with an initial vector $p^\circ \in \operatorname{dom} g$. Then, the number of updates of vector p is bounded by $\eta(p^\circ, p^*_{\min})$ in the up phase and by $2\eta(p^\circ, p^*_{\min})$ in the down phase; in total, bounded by $3\eta(p^\circ, p^*_{\min})$.*

By Theorem 1.1, we can improve the bound on the number of updates in the down phase. The behavior of TwoPhaseMinMin applied to an $\mathrm{L}^\natural$-convex function g is the same as that of TwoPhase applied to the $\mathrm{L}^\natural$-convex function $g_\varepsilon(p) = g(p) + \varepsilon \sum_{i=1}^n p(i)$ with a sufficiently small positive ε. Indeed, we have the following equivalences:

$$
\begin{aligned}
&X \subseteq N \text{ is a minimizer of } g_\varepsilon(p + \chi_X) \\
&\qquad \Longleftrightarrow X \text{ is the minimal minimizer of } g(p + \chi_X), \\
&X \subseteq N \text{ is a minimizer of } g_\varepsilon(p - \chi_X) \\
&\qquad \Longleftrightarrow X \text{ is the maximal minimizer of } g(p - \chi_X), \\
&p \in \mathbb{Z}^n_+ \text{ is a minimizer of } g_\varepsilon \\
&\qquad \Longleftrightarrow p \text{ is the minimal minimizer of } g.
\end{aligned}
$$

These facts, together with Theorem 1.1, imply the following bounds.

Theorem 1.4 *Suppose that the algorithm* TwoPhaseMinMin *is applied to an $L^\natural$-convex function g with an initial vector $p^\circ \in \operatorname{dom} g$. Then, the number of updates of vector p is bounded by $\eta(p^\circ, p^*_{\min})$ in the up phase and by $\eta(p^\circ, p^*_{\min})$ in the down phase; in total, bounded by $2\eta(p^\circ, p^*_{\min})$.*

2 Motivation from Auction Theory

This research is motivated by design and analysis of iterative auction in auction theory. In the auction literature an algorithm (a mechanism, more precisely) called iterative auction (also called dynamic auction, Walrasian tâtonnement process, etc.) is often used to find equilibrium prices of goods (see, e.g., [3,4]). An

[3] While the algorithm TwoPhaseMinMin in [13] is proposed as an algorithm for a specific $\mathrm{L}^\natural$-convex function (i.e., Lyapunov function), the algorithm as well as its analysis can be naturally extended to general $\mathrm{L}^\natural$-convex functions.

iterative auction updates prices repeatedly by using bidders' demand information, and finds equilibrium prices. A well-known iterative auction is the English auction for a single item.

Let us consider an auction market with n types of items, denoted by $N = \{1, 2, \ldots, n\}$, and m bidders, denoted by $M = \{1, 2, \ldots, m\}$. Each bidder $i \in M$ has his/her valuation function $f_i : 2^N \to \mathbb{Z}$ with the value $f_i(X)$ representing the degree of satisfaction for an item set $X \subseteq N$. We assume that each f_i is an integer-valued function satisfying the so-called "gross-substitutes" condition, which is a natural assumption for valuation functions (see [2,6,7] for the precise definition). An allocation of items is defined as a family of item sets $X_1, X_2, \ldots, X_m$ satisfying $X_i \cap X_h = \emptyset$ if $i \neq h$ and $\bigcup_{i \in M} X_i \subseteq N$.

The goal of an auction is to find equilibrium allocation and prices of items. A pair of price vector $p^* \in \mathbb{Z}^n_+$ and an allocation of items $X^*_1, X^*_2, \ldots, X^*_m$ is called a *Walrasian equilibrium* [3,4] if the following conditions hold:

$$X^*_i \in \arg\max\{f_i(X) - \sum_{j \in X} p(j) \mid X \subseteq N\} \qquad (i \in M),$$

$$p(j) = 0 \qquad (j \in N \setminus \bigcup_{i \in M} X^*_i).$$

Hence, in the equilibrium each bidder gets his/her best item set and all unsold items have zero price.

The natural and popular iterative auctions are ascending and descending auctions, in which prices are monotonically increasing or decreasing. Monotone movement of prices is preferable in iterative auctions since it makes easier to forecast the outcome of equilibrium price computation. For the *unit-demand* auction model where each bidder desires at most one item, an ascending auction and an descending auction are proposed by Demange–Gale–Sotomayor [5] and by Mishra–Parkes [9], respectively. These iterative auctions are generalized to the multi-demand auction model by Kelso–Crawford [7], Gul–Stacchetti [6], and Ausubel [2].

While ascending and descending auctions have a merit that the price movement is monotone, these iterative auctions have a drawback that the number of iterations is large. In an ascending auction, we cannot decrease prices during the computation, and therefore the initial prices should be lower bounds of unknown equilibrium prices. Therefore, even if the auctioneer knows the expectation of equilibrium prices, it is difficult to reduce the number of iterations by using the knowledge. It is customary in ascending auctions to set the initial prices to the lowest possible prices, but this makes the number of iterations large. This is also the case with descending auctions.

To make it possible to start from arbitrarily chosen prices, Andersson–Erlanson [1] proposed an iterative auction, for the unit-demand model, by combining the ascending auction by [5] and the descending auction by [9]. The algorithm admits the use of arbitrarily chosen initial prices, and consists of two phases, the price ascending phase and descending phase. Hence, the movement of prices is first monotone increasing and then monotone decreasing. Moreover, the flexibility in the choice of initial prices is useful in reducing the number of iterations, especially when the auctioneer has information about the expected equilibrium prices. Andersson–Erlanson [1] also theoretically analyzed the number of iterations in the two-phase auction algorithm.

The connection between equilibrium price computation and optimization is made clear in Ausubel [2], which shows that a price vector $p \in \mathbb{Z}^n_+$ is an equilibrium price vector if and only if p is a minimizer of the *Lyapunov function* $L : \mathbb{Z}^n_+ \to \mathbb{Z}$ given by

$$L(p) = \sum_{i=1}^{m} \max\{f_i(X) - p(X) \mid X \subseteq N\} + p(N) \quad (p \in \mathbb{Z}^n_+), \tag{2.4}$$

under the assumption that each f_i is an integer valued function satisfying the gross-substitutes condition. The connection to discrete convex analysis is pointed out in Murota–Shioura–Yang [12,13], which show that the Lyapunov function is an L$^\natural$-convex function, and hence iterative auctions can be seen as minimization algorithms for a specific L$^\natural$-convex function. Indeed, it is shown [13] that some of the existing iterative auctions coincide with L$^\natural$-convex function minimization algorithms applied to the Lyapunov function. In particular, the two-phase auction algorithm for the unit-demand model by Andersson–Erlanson [1] can be recognized as the algorithm TwoPhaseMinMin applied to the Lyapunov function. Moreover, for the multi-demand model, a two-phase auction algorithm is proposed in [12,13] (see also Ausubel [2]) by applying TwoPhaseMinMin to the Lyapunov function. Hence, our result (Theorem 1.4) provides tight bounds for the numbers of iterations required by the two-phase auction algorithms. We here rephrase Theorem 1.4 for the two-phase auction in [13].

Corollary 2.1 *For an arbitrarily chosen initial price vector $p^\circ \in \mathbb{Z}^n_+$, the two-phase auction of Murota–Shioura–Yang* [13] *terminates by outputting the unique minimal equilibrium price vector $p^*_{\rm min}$. The number of updates of vector p is bounded by $\eta(p^\circ,\ p^*_{\rm min})$ in the ascending phase and by $\eta(p^\circ,\ p^*_{\rm min})$ in the descending phase; in total, bounded by $2\eta(p^\circ,\ p^*_{\rm min})$.*

3 Proof

In this section we give a proof of Theorem 1.1 by improving the analysis of a more general minimization algorithm by Kolmogorov–Shioura [8] (named "primal algorithm" in [8]). In this algorithm, vector p can move upwards or downwards arbitrarily in each iteration, as far as the function value $g(p)$ decreases.

Algorithm GreedyUpDown
Step 0: Find a vector $p^\circ \in \mathrm{dom}\, g$ and set $p := p^\circ$.
    Set SuccessUp := false, SuccessDown := false.
Step 1: Do UP or DOWN in any order until SuccessUp = SuccessDown = true:
    UP (do only if SuccessUp is false):
        Find a minimizer $X \subseteq N$ of $g(p + \chi_X)$.
        If $g(p + \chi_X) = g(p)$, then set SuccessUp := true;
        otherwise set $p := p + \chi_X$.
    DOWN (do only if SuccessDown is false):
        Find a minimizer $X \subseteq N$ of $g(p - \chi_X)$.
        If $g(p - \chi_X) = g(p)$, then set SuccessDown := true;
        otherwise set $p := p - \chi_X$.
Step 2: Output p and stop.

The algorithm TwoPhase is a special implementation of the algorithm GreedyUpDown, where UP is performed repeatedly until SuccessUp is true, and then DOWN is performed repeatedly. We will show the following bounds for GreedyUpDown, where we say that an update of p in GreedyUpDown is an *up-update* if p is updated to $p + \chi_X$ with some nonempty X, and a *down-update* if p is updated to $p - \chi_X$ with some nonempty X.

Theorem 3.1 *Suppose that the algorithm* GreedyUpDown *is applied to an $L^\natural$-convex function g with an initial vector $p^\circ \in \mathrm{dom}\, g$. Then, the numbers of up-updates and down-updates of vector p are each bounded by $\mu(p^\circ)$; in total, the number of updates of p is bounded by $2\mu(p^\circ)$.*

Theorem 1.1 is an immediate corollary of this theorem.

In the following, we prove Theorem 3.1 by using the following property of $\mathrm{L}^\natural$-convex functions. For a vector $q \in \mathbb{Z}^n$, we denote $\mathrm{supp}^+(q) = \{j \in N \mid q(j) > 0\}$.

Lemma 3.2 ([10, Theorem 7.7]) *Let $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an $L^\natural$-convex function. For every $p, q \in \mathrm{dom}\, g$ with $\mathrm{supp}^+(p - q) \neq \emptyset$, it holds that*

$$g(p) + g(q) \geq g(p - \chi_Y) + g(q + \chi_Y)$$

with $Y = \arg\max_{j \in N}\{p(j) - q(j)\}$.

We denote by $p^{\oplus}$ the unique minimal vector in $\arg\min\{g(q) \mid q \in \mathbb{Z}^n,\ q \geq p^{\circ}\}$, and by $p^{\ominus}$ the unique maximal vector in $\arg\min\{g(q) \mid q \in \mathbb{Z}^n,\ q \leq p^{\circ}\}$. The number of updates of vector p in the algorithm GreedyUpDown is analyzed in [8].

Proposition 3.3 ([8, Section 2.2]) *The number of up-updates (resp., down-updates) of vector p in the algorithm* GreedyUpDown *is bounded by* $\|p^{\oplus} - p^{\circ}\|_{\infty}$ *(resp.,* $\|p^{\ominus} - p^{\circ}\|_{\infty}$*).*

The following lemma is a key observation to recast this result to Theorem 3.1.

Lemma 3.4 *It holds that* $\|p^{\oplus} - p^{\circ}\|_{\infty} \leq \mu(p^{\circ})$ *and* $\|p^{\ominus} - p^{\circ}\|_{\infty} \leq \mu(p^{\circ})$*.*

Proof We prove the former inequality $\|p^{\oplus} - p^{\circ}\|_{\infty} \leq \mu(p^{\circ})$ only since the latter inequality $\|p^{\ominus} - p^{\circ}\|_{\infty} \leq \mu(p^{\circ})$ can be proven in the same manner.

To prove the inequality $\|p^{\oplus} - p^{\circ}\|_{\infty} \leq \mu(p^{\circ})$, we show that $\|p^{\oplus} - p^{\circ}\|_{\infty} \leq \eta(p^*, p^{\circ})$ holds for every minimizer p^* of g; recall the definition of $\eta(p^*, p^{\circ})$ in (1.2).

We first consider the case with $p^* \geq p^{\circ}$. Since $p^{\oplus}$ is a minimizer of g in the set $\{q \in \mathbb{Z}^n \mid q \geq p^{\circ}\}$, we have $p^{\oplus} \in \arg\min g$. Hence, it holds that

$$\|p^{\oplus} - p^{\circ}\|_{\infty} \leq \|p^* - p^{\circ}\|_{\infty} = \eta(p^*, p^{\circ}).$$

We then assume that $\mathrm{supp}^+(p^{\circ} - p^*) \neq \emptyset$. This implies $\mathrm{supp}^+(p^{\oplus} - p^*) \neq \emptyset$. Let $Y = \arg\max_{j \in N}\{p^{\oplus}(j) - p^*(j)\}$.

Claim: There exists some $t \in Y$ such that $p^{\oplus}(t) = p^{\circ}(t)$.
[Proof of Claim] By Lemma 3.2, it holds that

$$g(p^{\oplus}) + g(p^*) \geq g(p^{\oplus} - \chi_Y) + g(p^* + \chi_Y). \tag{3.5}$$

Since p^* is a minimizer of g, we have $g(p^* + \chi_Y) \geq g(p^*)$, which, combined with (3.5), implies $g(p^{\oplus} - \chi_Y) \leq g(p^{\oplus})$. From this inequality follows that $p^{\oplus} - \chi_Y \not\geq p^{\circ}$ since $p^{\oplus}$ is the minimal vector in the set $\arg\min\{g(q) \mid q \in \mathbb{Z}^n,\ q \geq p^{\circ}\}$. Hence, there exists some $t \in Y$ with $p^{\oplus}(t) = p^{\circ}(t)$. [End of Proof of Claim]

It holds that

$$\begin{aligned}
\|p^* - p^{\circ}\|_{\infty}^{-} &= \max_{j \in N}\{p^{\circ}(j) - p^*(j)\} && (\text{by } \mathrm{supp}^+(p^{\circ} - p^*) \neq \emptyset) \\
&\geq p^{\circ}(t) - p^*(t) \\
&= p^{\oplus}(t) - p^*(t) && (\text{by the claim above}) \\
&= \max_{j \in N}\{p^{\oplus}(j) - p^*(j)\} && (\text{by } t \in Y \text{ and the definition of } Y).
\end{aligned}$$

It follows that for every $k \in N$, we have

$$\begin{aligned}
p^{\oplus}(k) - p^{\circ}(k) &= [p^*(k) - p^{\circ}(k)] + [p^{\oplus}(k) - p^*(k)] \\
&\leq \|p^* - p^{\circ}\|_{\infty}^{+} + \max_{j \in N}\{p^{\oplus}(j) - p^*(j)\} \\
&= \|p^* - p^{\circ}\|_{\infty}^{+} + \|p^* - p^{\circ}\|_{\infty}^{-} = \eta(p^*, p^{\circ}).
\end{aligned}$$

Hence, $\|p^{\oplus} - p^{\circ}\|_{\infty} \leq \eta(p^*, p^{\circ})$ holds. □

Theorem 3.1 follows from Proposition 3.3 and Lemma 3.4.

4 Concluding Remark

In this note we analyzed the number of updates of vector p in the algorithm TwoPhase by using the η-distance $\mu(p^\circ)$ from the initial vector p° to the nearest minimizer of the given $\mathrm{L}^\natural$-convex function g. In [11,12] some "monotone" algorithms for $\mathrm{L}^\natural$-convex function minimization, where vector p is monotone increasing or decreasing, are analyzed, and the bounds for the numbers of updates of vector p are obtained in terms of the following distances:

$$\hat{\mu}(p^\circ) = \min\{\|p^* - p^\circ\|_\infty \mid p^* \in \arg\min g,\ p^* \geq p^\circ\} \qquad (p^\circ \in \mathbb{Z}^n),$$
$$\check{\mu}(p^\circ) = \min\{\|p^* - p^\circ\|_\infty \mid p^* \in \arg\min g,\ p^* \leq p^\circ\} \qquad (p^\circ \in \mathbb{Z}^n),$$

where it is assumed that $\hat{\mu}(p^\circ) = +\infty$ (resp., $\check{\mu}(p^\circ) = +\infty$) if $\arg\min g \cap \{q \in \mathbb{Z}^n \mid q \geq p^\circ\} = \emptyset$ (resp., $\arg\min g \cap \{q \in \mathbb{Z}^n \mid q \leq p^\circ\} = \emptyset$).

By definition, we have $\hat{\mu}(p^\circ) \geq \mu(p^\circ)$ and $\check{\mu}(p^\circ) \geq \mu(p^\circ)$ for every $p^\circ \in \mathbb{Z}^n$. In fact, the analysis in Section 3 can be used to show that the inequalities hold with equality whenever $\hat{\mu}(p^\circ)$ and $\check{\mu}(p^\circ)$ take finite values.

Proposition 4.1
(i) $\hat{\mu}(p^\circ) = \mu(p^\circ)$ *holds if* $\arg\min g \cap \{q \in \mathbb{Z}^n \mid q \geq p^\circ\} \neq \emptyset$
(ii) $\check{\mu}(p^\circ) = \mu(p^\circ)$ *holds if* $\arg\min g \cap \{q \in \mathbb{Z}^n \mid q \leq p^\circ\} \neq \emptyset$.

Proof We prove (i) only since (ii) can be proved similarly. It suffices to show that $\hat{\mu}(p^\circ) \leq \mu(p^\circ)$. Since $\arg\min g \cap \{q \in \mathbb{Z}^n \mid q \geq p^\circ\} \neq \emptyset$, we have $p^\oplus \in \arg\min g$, where $p^\oplus$ is defined in Section 3. By the first inequality in Lemma 3.4, we have $\hat{\mu}(p^\circ) \leq \|p^\oplus - p^\circ\|_\infty \leq \mu(p^\circ)$. □

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